

Gauge Theory of Gravity

Ning Wu *

Institute of High Energy Physics, P.O.Box 918-1, Beijing 100039, P.R.China †

February 1, 2008

PACS Numbers: 11.15.-q, 04.60.-m, 11.10.Gh.

Keywords: gauge field, quantum gravity, renormalization.

Abstract

The quantum gravity is formulated based on principle of local gauge invariance. The model discussed in this paper has local gravitational gauge symmetry and gravitational field is represented by gauge field. Path integral quantization of the theory is discussed in the paper. A strict proof on the renormalizability of the theory is also given. In leading order approximation, the gravitational gauge field theory gives out classical Newton's theory of gravity. In first order approximation and for vacuum, the gravitational gauge field theory gives out Einstein's general theory of relativity. Using this new quantum gauge theory of gravity, we can explain some important puzzles of Nature. This quantum gauge theory of gravity is a renormalizable quantum theory.

*email address: wuning@heli.ihep.ac.cn

†mailing address

1 Introduction

Gravity is an ancient topic in science. In ancient times, human has known the existence of weight. Now we know that it is the gravity between an object and earth. In 1686, Isaac Newton published his epoch-making book *MATHEMATICAL PRINCIPLES OF NATURAL PHILOSOPHY*. In this book, through studying the motion of planet in solar system, he found that gravity obeys the inverse square law[1]. The Newton's classical theory of gravity is kept unchanged until 1916. At that year, Einstein published his epoch-making paper on General Relativity[2, 3]. In this great work, he founded a relativistic theory on gravity, which is based on principle of general relativity and equivalence principle. Newton's classical theory for gravity appears as a classical limit of general relativity.

One of the biggest revolution in human kind in the last century is the foundation of quantum theory. The quantum hypothesis was first introduced into physics by Max Plank in 1900. Inspired by his quantum hypothesis, Albert Einstein used it to explain the photoelectric effect successfully and Niels Bohr used it to explain the positions of spectral lines of Hydrogen. In 1923, Louis de Broglie proposed the principle of wave-particle duality. In the years 1925-1926, Werner Heisenberg, Max Born, Pascual Jordan and Wolfgang Pauli develop matrix mechanics, and in 1926, Erwin Schroedinger develop wave mechanics. Soon after, relativistic quantum mechanics and quantum field theory are proposed by P.A.M.Dirac, Oskar Klein, Walter Gordan and others.

In 1921, H.Weyl introduced the concept of gauge transformation into physics[4, 5], which is one of the most important concepts in modern physics, though his original theory is not successful. Later, V.Fock, H.Weyl and W.Pauli found that quantum electrodynamics is a gauge invariant theory[6, 7, 8]. In 1954, Yang and Mills proposed non-Abel gauge field theory[9]. This theory was soon applied to elementary particle physics. Unified electroweak theory[10, 11, 12] and quantum chromodynamics are all based on gauge field theory. The predictions of unified electroweak theory have been confirmed in a large number of experiments, and the intermediate gauge bosons W^\pm and Z^0 which are predicted by unified electroweak model are also found in experiments. The $U(1)$ part of the unified electroweak model, quantum electrodynamics, now become one of the most accurate and best-tested theories of modern physics. All these achievements of gauge field theories suggest that gauge field theory is a fundamental theory that describes fundamental interactions. Now, it is generally believed that four kinds of fundamental interactions in Nature are all gauge interactions and they can be described by gauge field theory. From theoretical point of view, the principle of local gauge invariance plays a fundamental role in particle's interaction theory.

In 1916, Albert Einstein points out that quantum effects must lead to modifications in the theory of general relativity[13]. Soon after the foundation of quantum mechanics, physicists try to found a theory that could describe the quantum behavior of the full gravitational field. In the 70 years attempts, physicists have found two theories based on quantum mechanics that attempt to unify general relativity and quantum mechanics, one is canonical quantum gravity and another is superstring theory. But for quantum field theory, there are different kinds of mathematical infinities that naturally occur in quantum descriptions of fields. These infinities should be removed by the technique of perturbative renormalization. However, the perturbative renormalization does not work for the quantization of Einstein's theory of gravity, especially in canonical quantum gravity. In superstring theory, in order to make perturbative renormalization to work, physicists have to introduce six extra dimensions. But up to now, none of the extra dimensions have been observed. To found a consistent theory that can unify general relativity and quantum mechanics is a long dream for physicists.

The "relativity revolution" and the "quantum revolution" are among the greatest successes of twentieth century physics, yet two theories appears to be fundamentally incompatible. General relativity remains a purely classical theory which describes the geometry of space and time as smooth and continuous, on the contrary, quantum mechanics divides everything into discrete quanta. The underlying theoretical incompatibility between two theories arises from the way that they treat the geometry of space and time. This situation makes some physicists still wonder whether quantum theory is a truly fundamental theory of Nature, or just a convenient description of some aspects of the microscopic world. Some physicists even consider the twentieth century as the century of the incomplete revolution. To set up a consistent quantum theory of gravity is considered to be the last challenge of quantum theory[14, 15]. In other words, combining general relativity with quantum mechanics is considered to be the last hurdle to be overcome in the "quantum revolution".

Gauge treatment of gravity was suggested immediately after the gauge theory birth itself[16, 17, 18]. In the traditional gauge treatment of gravity, Lorentz group is localized, and the gravitational field is not represented by gauge potential[19, 20, 21]. It is represented by metric field. (In this paper, we will find that metric field and gauge field are not independent. We can use gauge field to determine metric field.) The theory has beautiful mathematical forms, but up to now, its renormalizability is not proved. In other words, it is conventionally considered to be non-renormalizable.

I will not talk too much on the history of quantum gravity and the incompatibilities between quantum mechanics and general relativity here. Materials on these

subject can be widely found in literatures. Now we want to ask that, except for traditional canonical quantum gravity and superstring theory, whether exists another approach to set up a fundamental theory, in which general relativity and quantum mechanics are compatible.

Recently, some new attempts were proposed to use Yang-Mills theory to reformulate quantum gravity[22, 23, 24, 25]. In these new approaches, the importance of gauge fields is emphasized. Some physicists also try to use gauge potential to represent gravitational field, some suggest that we should pay more attention on translation group. In this paper, a completely new attempt is proposed. In other words, we try to use completely new notions and completely new methods to formulate quantum gravity. Our goal is try to set up a renormalizable quantum gauge theory of gravity. Its relation to the traditional quantum gravity is not studied in this paper, which is an independent work that will be done in the near future. Maybe this new formulation of quantum gravity is equivalent to the traditional formulation, maybe they are not equivalent. The main goal that we hope to formulate this new quantum gauge theory of gravity is not to deny traditional quantum gravity, but to prove the renormalizability of quantum gravity, for the renormalizability of quantum gravity is easy to be proved in this new formulation. It is known that the mathematical formulation of traditional quantum gravity is quite different from that of the new quantum gauge theory of gravity which is formulated in this paper, but it does not mean that they are essentially different in physics. In a meaning, maybe the present situation is somewhat similar to that of the quantum mechanics. It is well know that quantum mechanics can be formulated in different representations, such as Schrodinger representation, Heisenberg representation, \dots . Though mathematical formulations of quantum mechanics are quite different in different representations, they are essentially the same in physics. Quantum gravity belongs to quantum mechanics, we can imagine that it will of course have many different representations. If some people can prove that the traditional quantum gravity is equivalent to this new renormalizable quantum gauge theory of gravity, then the renormalizability of the traditional quantum gravity will automatically hold. If this is true, then two kinds of quantum gravity can be regarded as two different representations of quantum gravity. Collaborating with Prof. Zhan Xu and Prof. Dahua Zhang, we have found the differential geometrical formulation of the gravitational gauge theory of gravity which is formulated in this paper[26]. The relation between traditional quantum gravity and gravitational gauge theory of gravity is under studying now.

As we have mentioned above, gauge field theory provides a fundamental tool to study fundamental interactions. In this paper, we will use this tool to study quantum gravity. We will use a completely new language to express the quantum theory of gravity. In order to do this, we first need to introduce some transcendental foun-

dations of this new theory, which is the most important thing to formulate the whole theory. Then we will discuss a new kind of non-Abel gauge group, which will be the fundamental symmetry of quantum gravity. For the sake of simplicity, we call this group gravitational gauge group. After that, we will construct a Lagrangian which has local gravitational gauge symmetry. In this Lagrangian, gravitational field appears as the gauge field of the gravitational gauge symmetry. Then we will discuss the gravitational interactions between scalar field (or Dirac field or vector field) and gravitational field. Just as what Albert Einstein had ever said in 1916 that quantum effects must lead to modifications in the theory of general relativity, there are indeed quantum modifications in this new quantum gauge theory of gravity. In other words, the local gravitational gauge symmetry requires some additional interaction terms other than those given by general relativity. This new quantum theory of gravity can even give out an exact relationship between gravitational fields and space-time metric in generally relativity. The classical limit of this new quantum theory will give out classical Newton's theory of gravity and general relativity. In other words, the leading order approximation of the new theory gives out classical Newton's theory of gravity, the first order approximation of the new theory gives out Einstein's general theory of relativity. This can be regarded as the first test of the new theory. Then we will discuss quantization of gravitational gauge field. Something most important is that this new quantum theory of gravity is a renormalizable theory. A formal strict proof on the renormalizability of this new quantum theory of gravity is given in this paper. After that, we will discuss some theoretical predictions of this new quantum theory of gravity. In this chapter, we will find that some puzzles which can not be explained in traditional theory can be explained by this new quantum gauge theory of gravity. These explanations can be regarded as the second test of the new theory. I hope that the effort made in this paper will be beneficial to our understanding on the quantum aspects of gravitational field. The relationship between this new quantum theory of gravity and traditional canonical quantum gravity or superstring theory is not study now, and I hope that this work will be done in the near future. Because the new quantum theory of gravity is logically independent of traditional quantum gravity, we need not discuss traditional quantum gravity first. Anyone who is familiar with traditional non-Abel gauge field theory can understand the whole paper. In other words, readers who never study anything on traditional quantum gravity can understand this new quantum theory of gravity. Now, let's begin our long journey to the realm of logos.

2 The Transcendental Foundations

It is known that action principle is one of the most important fundamental principle in quantum field theory. Action principle says that any quantum system is described by an action. The action of the system contains all interaction information, contains all information of the fundamental dynamics. The least of the action gives out the classical equation of motion of a field. Action principle is widely used in quantum field theory. We will accept it as one of the most fundamental principles in this new quantum theory of gravity. The rationality of action principle will not be discussed here, but it is well know that the rationality of the action principle has already been tested by a huge amount of experiments. However, this principle is not a special principle for quantum gravity, it is a ubiquitous principle in quantum field theory. Quantum gravity discussed in this paper is a kind of quantum field theory, it's naturally to accept action principle as one of its fundamental principles.

We need a special fundamental principle to introduce quantum gravitational field, which should be the foundation of all kinds of fundamental interactions in Nature. This special principle is gauge principle. In order to introduce this important principle, let's first study some fundamental laws in some fundamental interactions other than gravitational interactions. We know that, except for gravitational interactions, there are strong interactions, electromagnetic interactions and weak interactions, which are described by quantum chromodynamics, quantum electrodynamics and unified electroweak theory respectively. Let's study these three fundamental interactions one by one.

Quantum electrodynamics (QED) is one of the most successful theory in physics which has been tested by most accurate experiments. Let's study some logic in QED. It is know that QED theory has $U(1)$ gauge symmetry. According to Noether's theorem, there is a conserved charge corresponding to the global $U(1)$ gauge transformations. This conserved charge is just the ordinary electric charge. On the other hand, in order to keep local $U(1)$ gauge symmetry of the system, we had to introduce a $U(1)$ gauge field, which transmits electromagnetic interactions. This $U(1)$ gauge field is just the well-know electromagnetic field. The electromagnetic interactions between charged particles and the dynamics of electromagnetic field are completely determined by the requirement of local $U(1)$ gauge symmetry. The source of this electromagnetic field is just the conserved charge which is given by Noether's theorem. After quantization of the field, this conserved charge becomes the generator of the quantum $U(1)$ gauge transformations. The quantum $U(1)$ gauge transformation has only one generator, it has no generator other than the quantum electric charge.

Quantum chromodynamics (QCD) is a prospective fundamental theory for strong

interactions. QCD theory has $SU(3)$ gauge symmetry. The global $SU(3)$ gauge symmetry of the system gives out conserved charges of the theory, which are usually called color charges. The local $SU(3)$ gauge symmetry of the system requires introduction of a set of $SU(3)$ non-Abel gauge fields, and the dynamics of non-Abel gauge fields and the strong interactions between color charged particles and gauge fields are completely determined by the requirement of local $SU(3)$ gauge symmetry of the system. These $SU(3)$ non-Abel gauge fields are usually call gluon fields. The sources of gluon fields are color charges. After quantization, these color charges become generators of quantum $SU(3)$ gauge transformation. Something which is different from $U(1)$ Abel gauge symmetry is that gauge fields themselves carry color charges.

Unified electroweak model is the fundamental theory for electroweak interactions. Unified electroweak model is usually called the standard model. It has $SU(2)_L \times U(1)_Y$ symmetry. The global $SU(2)_L \times U(1)_Y$ gauge symmetry of the system also gives out conserved charges of the system, The requirement of local $SU(2)_L \times U(1)_Y$ gauge symmetry needs introducing a set of $SU(2)$ non-Abel gauge fields and one $U(1)$ Abel gauge field. These gauge fields transmit weak interactions and electromagnetic interactions, which correspond to intermediate gauge bosons W^\pm , Z^0 and photon. The sources of these gauge fields are just the conserved Noether charges. After quantization, these conserved charges become generators of quantum $SU(2)_L \times U(1)_Y$ gauge transformation.

QED, QCD and the standard model are three fundamental theories of three kinds of fundamental interactions. Now we want to summarize some fundamental laws of Nature on interactions. Let's first ruminant over above discussions. Then we will find that our formulations on three different fundamental interaction theories are almost completely the same, that is the global gauge symmetry of the system gives out conserved Noether charges, in order to keep local gauge symmetry of the system, we have to introduce gauge field or a set of gauge fields, these gauge fields transmit interactions, and the source of these gauge fields are the conserved charges and these conserved Noether charges become generators of quantum gauge transformations after quantization. These will be the main content of gauge principle.

Before we formulate gauge principle formally, we need to study something more on symmetry. It is know that not all symmetries can be localized, and not all symmetries can be regarded as gauge symmetries and have corresponding gauge fields. For example, time reversal symmetry, space reflection symmetry, \dots are those kinds of symmetries. We can not find any gauge fields or interactions which correspond to these symmetries. It suggests that symmetries can be divided into two different classes in nature. Gauge symmetry is a special kind of symmetry which has the fol-

lowing properties: 1) it can be localized; 2) it has some conserved charges related to it; 3) it has a kind of interactions related to it; 4) it is usually a continuous symmetry. This symmetry can completely determine the dynamical behavior of a kind of fundamental interactions. For the sake of simplicity, we call this kind of symmetry dynamical symmetry or gauge symmetry. Any kind of fundamental interactions has a gauge symmetry corresponding to it. In QED, the $U(1)$ symmetry is a gauge symmetry, in QCD, the color $SU(3)$ symmetry is a gauge symmetry and in the standard model, the $SU(2)_L \times U(1)_Y$ symmetry is also a gauge symmetry. The gravitational gauge symmetry which we will discuss below is also a kind of gauge symmetry. The time reversal symmetry and space reflection symmetry are not gauge symmetries. Those global symmetries which can not be localized are not gauge symmetries either. Gauge symmetry is a fundamental concept for gauge principle.

Gauge principle can be formulated as follows: Any kind of fundamental interactions has a gauge symmetry corresponding to it; the gauge symmetry completely determines the forms of interactions. In principle, the gauge principle has the following four different contents:

1. **Conservation Law:** the global gauge symmetry gives out conserved current and conserved charge;
2. **Interactions:** the requirement of the local gauge symmetry requires introduction of gauge field or a set of gauge fields; the interactions between gauge fields and matter fields are completely determined by the requirement of local gauge symmetry; these gauge fields transmit the corresponding interactions;
3. **Source:** qualitative speaking, the conserved charge given by global gauge symmetry is the source of gauge field; for non-Abel gauge field, gauge field is also the source of itself;
4. **Quantum Transformation:** the conserved charges given by global gauge symmetry become generators of quantum gauge transformation after quantization, and for this kind of interactions, the quantum transformation can not have generators other than quantum conserved charges given by global gauge symmetry.

It is known that conservation law is the objective origin of gauge symmetry, so gauge symmetry is the exterior exhibition of the interior conservation law. The conservation law is the law that exists in fundamental interactions, so fundamental interactions are the logic precondition and foundation of the conservation law. Gauge principle tells us how to study conservation law and fundamental interactions through symmetry. Gauge principle is one of the most important transcendental fundamental principles for all kinds of fundamental interactions in Nature; it reveals

the common nature of all kinds of fundamental interactions in Nature. It is also the transcendental foundation of the quantum gravity which is formulated in this paper. It will help us to select the gauge symmetry for quantum gravitational theory and help us to determine the Lagrangian of the system. In a meaning, we can say that without gauge principle, we can not set up this new renormalizable quantum gauge theory of gravity.

Another transcendental principle that widely used in quantum field theory is the microscopic causality principle. The central idea of the causality principle is that any changes in the objective world have their causation. Quantum field theory is a relativistic theory. It is know that, in the special theory of relativity, the limit spread speed is the speed of light. It means that, in a definite reference system, the limit spread speed of the causation of some changes is the speed of light. Therefore, the special theory of relativity exclude the possibility of the existence of any kinds of non-local interactions in a fundamental theory. Quantum field theory inherits this basic idea and calls it the microscopic causality principle. There are several expressions of the microscopic causality principle in quantum field theory. One expression say that two events which happen at the same time but in different space position are two independent events. The mathematical formulation for microscopic causality principle is that

$$[O_1(\vec{x}, t) , O_2(\vec{y}, t)] = 0, \quad (2.1)$$

when $\vec{x} \neq \vec{y}$. In the above relation, $O_1(\vec{x}, t)$ and $O_2(\vec{y}, t)$ are two different arbitrary local bosonic operators. Another important expression of the microscopic causality principle is that, in the Lagrangian of a fundamental theory, all operators appear in the same point of space-time. Gravitational interactions are a kind of physical interactions, the fundamental theory of gravity should also obey microscopic causality principle. This requirement is realized in the construction of the Lagrangian for gravity. We will require that all field operators in the Lagrangian should be at the same point of space-time.

Because quantum field theory is a kind of relativistic theory, it should obey some fundamental principles of the special theory of relativity, such as principle of special relativity and principle of invariance of light speed. These two principles conventionally exhibit themselves through Lorentz invariance. So, in constructing the Lagrangian of the quantum theory of gravity, we require that it should have Lorentz invariance. This is also a transcendental requirement for the quantum theory of gravity. But what we treat here that is different from that of general relativity is that we do not localize Lorentz transformation. Because gauge principle forbids us to localize Lorentz transformation, asks us only to localize gravitational gauge transformation. We will discuss this topic in details later. However, it is important

to remember that global Lorentz invariance of the Lagrangian is a fundamental requirement. The requirement of global Lorentz invariance can also be treated as a transcendental principle of the quantum theory of gravity.

It is well-known that two transcendental principles of general relativity are principle of general relativity and principle of equivalence. It should be stated that, in the new gauge theory of gravity, the principle of general relativity appears in another way, that is, it realized itself through local gravitational gauge symmetry. From mathematical point of view, the local gravitational gauge invariance is just the general covariance in general relativity. In the new quantum theory of gravity, principle of equivalence plays no role. In other words, we will not accept principle of equivalence as a transcendental principle of the new quantum theory of gravity, for gauge principle is enough for us to construct quantum theory of gravity. We will discuss something more about principle of equivalence later.

3 Gravitational Gauge Group

Before we start our mathematical formulation of gravitational gauge theory, we have to determine which group is the gravitational gauge group, which is the starting point of the whole theory. It is know that, in the traditional quantum gauge theory of gravity, Lorentz group is localized. We will not follow this way, for it contradicts with gauge principle. Now, we use gauge principle to determine which group is the exact group for gravitational gauge theory.

Some of the most important properties of gravity can be seen from Newton's classical gravity. In this classical theory of gravity, gravitational force between two point objects is given by:

$$f = G \frac{m_1 m_2}{r^2} \quad (3.1)$$

with m_1 and m_2 masses of two objects, r the distance between two objects. So, gravity is proportional to the masses of both objects, in other words, mass is the source of gravitational field. In general relativity, Einstein's gravitational equation is the equation which gives out the relation between energy-momentum tensor and space-time curvature, which is essentially the relation between energy-momentum tensor and gravitational field. In the Einstein's gravitational equation, energy-momentum is treated as the source of gravity. This opinion is inherited in the new quantum theory gauge of gravity. In other words, the starting point of the new quantum gauge theory of gravity is that the energy-momentum is the source of gravitational field. According to rule 3 and rule 1 of gauge principle, we know that, energy-momentum is the conserved charges of the corresponding global symmetry, which is just the

symmetry for gravity. According to quantum field theory, energy-momentum is the conserved charge of global space-time translation, the corresponding conserved current is energy-momentum tensor. Therefore, the global space-time translation is the global gravitational gauge transformation. According to rule 4, we know that, after quantization, the energy-momentum operator becomes the generator of gravitational gauge transformation. It also states that, except for energy-momentum operator, there is no other generator for gravitational gauge transformation, such as, angular momentum operator $M_{\mu\nu}$ can not be the generator of gravitational gauge transformation. This is the reason why we do not localize Lorentz transformation in this new quantum gauge theory of gravity, for the generator of Lorentz transformation is not energy-momentum operator. According to rule 2 of gauge principle, the gravitational interactions will be completely determined by the requirement of the local gravitational gauge symmetry. These are the basic ideas of the new quantum gauge theory of gravity, and they are completely deductions of gauge principle.

We know that the generator of Lorentz group is angular momentum operator $M_{\mu\nu}$. If we localize Lorentz group, according to gauge principle, angular momentum will become source of a new field, which transmits direct spin interactions. This kind of interactions does not belong to traditional Newton-Einstein gravity. It is a new kind of interactions. Up to now, we do not know that whether this kind of interactions exists in Nature or not. Besides, spin-spin interaction is a kind of non-renormalizable interaction. In other words, a quantum theory which contains spin-spin interaction is a non-renormalizable quantum theory. For these reasons, we will not localize Lorentz group in this paper. We only localize translation group in this paper. We will find that go along this way, we can set up a consistent quantum gauge theory of gravity which is renormalizable. In other words, only localizing space-time translation group is enough for us to set up a consistent quantum gravity.

From above discussions, we know that, from mathematical point of view, gravitational gauge transformation is the inverse transformation of space-time translation, and gravitational gauge group is space-time translation group. Suppose that there is an arbitrary function $\phi(x)$ of space-time coordinates x^μ . The global space-time translation is:

$$x^\mu \rightarrow x'^\mu = x^\mu + \epsilon^\mu. \quad (3.2)$$

The corresponding transformation for function $\phi(x)$ is

$$\phi(x) \rightarrow \phi'(x') = \phi(x) = \phi(x' - \epsilon). \quad (3.3)$$

According to Taylor series expansion, we have:

$$\phi(x - \epsilon) = \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \epsilon^{\mu_1} \cdots \epsilon^{\mu_n} \partial_{\mu_1} \cdots \partial_{\mu_n} \right) \phi(x), \quad (3.4)$$

where

$$\partial_{\mu_i} = \frac{\partial}{\partial x^{\mu_i}}. \quad (3.5)$$

Let's define a special exponential operation here. Define

$$E^{a^\mu \cdot b_\mu} \triangleq 1 + \sum_{n=1}^{\infty} \frac{1}{n!} a^{\mu_1} \cdots a^{\mu_n} \cdot b_{\mu_1} \cdots b_{\mu_n}. \quad (3.6)$$

This definition is quite different from that of ordinary exponential function. In general cases, operators a^μ and b_μ do not commute each other, so

$$E^{a^\mu \cdot b_\mu} \neq E^{b_\mu \cdot a^\mu}, \quad (3.7)$$

$$E^{a^\mu \cdot b_\mu} \neq e^{a^\mu \cdot b_\mu}, \quad (3.8)$$

where $e^{a^\mu \cdot b_\mu}$ is the ordinary exponential function whose definition is

$$e^{a^\mu \cdot b_\mu} \equiv 1 + \sum_{n=1}^{\infty} \frac{1}{n!} (a^{\mu_1} \cdot b_{\mu_1}) \cdots (a_{\mu_n} \cdot b_{\mu_n}). \quad (3.9)$$

If operators a^μ and b_μ commute each other, we will have

$$E^{a^\mu \cdot b_\mu} = E^{b_\mu \cdot a^\mu}, \quad (3.10)$$

$$E^{a^\mu \cdot b_\mu} = e^{a^\mu \cdot b_\mu}. \quad (3.11)$$

The translation operator \hat{U}_ϵ can be defined by

$$\hat{U}_\epsilon \equiv 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \epsilon^{\mu_1} \cdots \epsilon^{\mu_n} \partial_{\mu_1} \cdots \partial_{\mu_n}. \quad (3.12)$$

Then we have

$$\phi(x - \epsilon) = (\hat{U}_\epsilon \phi(x)). \quad (3.13)$$

In order to have a good form which is similar to ordinary gauge transformation operators, the form of \hat{U}_ϵ can also be written as

$$\hat{U}_\epsilon = E^{-i\epsilon^\mu \cdot \hat{P}_\mu}, \quad (3.14)$$

where

$$\hat{P}_\mu = -i \frac{\partial}{\partial x^\mu}. \quad (3.15)$$

\hat{P}_μ is just the energy-momentum operator in space-time coordinate space. In the definition of \hat{U}_ϵ of eq.(3.14), ϵ^μ can be independent of space-time coordinate or a function of space-time coordinate, in a word, it can be any functions of space time coordinate x .

Some operation properties of translation operator \hat{U}_ϵ are summarized below.

1. Operator \hat{U}_ϵ translate the space-time point of a field from x to $x - \epsilon$,

$$\phi(x - \epsilon) = (\hat{U}_\epsilon \phi(x)), \quad (3.16)$$

where ϵ^μ can be any function of space-time coordinate. This relation can also be regarded as the definition of the translation operator \hat{U}_ϵ .

2. If ϵ is a function of space-time coordinate, that is $\partial_\mu \epsilon^\nu \neq 0$, then

$$\hat{U}_\epsilon = E^{-i\epsilon^\mu \cdot \hat{P}_\mu} \neq E^{-i\hat{P}_\mu \cdot \epsilon^\mu}, \quad (3.17)$$

and

$$\hat{U}_\epsilon = E^{-i\epsilon^\mu \cdot \hat{P}_\mu} \neq e^{-i\epsilon^\mu \cdot \hat{P}_\mu}. \quad (3.18)$$

If ϵ is a constant, that is $\partial_\mu \epsilon^\nu = 0$, then

$$\hat{U}_\epsilon = E^{-i\epsilon^\mu \cdot \hat{P}_\mu} = E^{-i\hat{P}_\mu \cdot \epsilon^\mu}, \quad (3.19)$$

and

$$\hat{U}_\epsilon = E^{-i\epsilon^\mu \cdot \hat{P}_\mu} = e^{-i\epsilon^\mu \cdot \hat{P}_\mu}. \quad (3.20)$$

3. Suppose that $\phi_1(x)$ and $\phi_2(x)$ are two arbitrary functions of space-time coordinate, then we have

$$\left(\hat{U}_\epsilon(\phi_1(x) \cdot \phi_2(x)) \right) = (\hat{U}_\epsilon \phi_1(x)) \cdot (\hat{U}_\epsilon \phi_2(x)) \quad (3.21)$$

4. Suppose that A^μ and B_μ are two arbitrary operators in Hilbert space, λ is an arbitrary ordinary c-number which is commutate with operators A^μ and B_μ , then we have

$$\frac{d}{d\lambda} E^{\lambda A^\mu \cdot B_\mu} = A^\mu \cdot E^{\lambda A^\mu \cdot B_\mu} \cdot B_\mu. \quad (3.22)$$

5. Suppose that ϵ is an arbitrary function of space-time coordinate, then

$$(\partial_\mu \hat{U}_\epsilon) = -i(\partial_\mu \epsilon^\nu) \hat{U}_\epsilon \hat{P}_\nu. \quad (3.23)$$

6. Suppose that A^μ and B_μ are two arbitrary operators in Hilbert space, then

$$tr(E^{A^\mu \cdot B_\mu} E^{-B_\mu \cdot A^\mu}) = tr \mathbf{I}, \quad (3.24)$$

where tr is the trace operation and \mathbf{I} is the unit operator in the Hilbert space.

7. Suppose that A^μ , B_μ and C^μ are three operators in Hilbert space, but operators A^μ and C^ν commutate each other:

$$[A^\mu, C^\nu] = 0, \quad (3.25)$$

then

$$tr(E^{A^\mu \cdot B_\mu} E^{B_\nu \cdot C^\nu}) = tr(E^{(A^\mu + C^\mu) \cdot B_\mu}). \quad (3.26)$$

8. Suppose that A^μ , B_μ and C^ν are three operators in Hilbert space, they satisfy

$$\begin{aligned} [A^\mu, C^\nu] &= 0, \\ [B_\mu, C^\nu] &= 0, \end{aligned} \quad (3.27)$$

then

$$E^{A^\mu \cdot B_\mu} E^{C^\nu \cdot B_\nu} = E^{(A^\mu + C^\mu) \cdot B_\mu}. \quad (3.28)$$

9. Suppose that A^μ , B_μ and C^ν are three operators in Hilbert space, they satisfy

$$\begin{aligned} [A^\mu, C^\nu] &= 0, \\ [[B_\mu, C^\nu], A^\rho] &= 0, \\ [[B_\mu, C^\nu], C^\rho] &= 0, \end{aligned} \quad (3.29)$$

then,

$$E^{A^\mu \cdot B_\mu} E^{C^\nu \cdot B_\nu} = E^{(A^\mu + C^\mu) \cdot B_\mu} + [E^{A^\mu \cdot B_\mu}, C^\sigma] E^{C^\nu \cdot B_\nu} B_\sigma. \quad (3.30)$$

10. Suppose that \hat{U}_{ϵ_1} and \hat{U}_{ϵ_2} are two arbitrary translation operators, define

$$\hat{U}_{\epsilon_3} = \hat{U}_{\epsilon_2} \cdot \hat{U}_{\epsilon_1}, \quad (3.31)$$

then,

$$\epsilon_3^\mu(x) = \epsilon_2^\mu(x) + \epsilon_1^\mu(x - \epsilon_2(x)). \quad (3.32)$$

This property means that the product to two translation operator satisfy closure property, which is one of the conditions that any group must satisfy.

11. Suppose that \hat{U}_ϵ is a non-singular translation operator, then

$$\hat{U}_\epsilon^{-1} = E^{i\epsilon^\mu(f(x)) \cdot \hat{P}_\mu}, \quad (3.33)$$

where $f(x)$ is defined by the following relations:

$$f(x - \epsilon(x)) = x. \quad (3.34)$$

\hat{U}_ϵ^{-1} is the inverse operator of \hat{U}_ϵ , so

$$\hat{U}_\epsilon^{-1} \hat{U}_\epsilon = \hat{U}_\epsilon \hat{U}_\epsilon^{-1} = \mathbf{1}, \quad (3.35)$$

where $\mathbf{1}$ is the unit element of the gravitational gauge group.

12. The product operation of translation also satisfies associative law. Suppose that \hat{U}_{ϵ_1} , \hat{U}_{ϵ_2} and \hat{U}_{ϵ_3} are three arbitrary translation operators, then

$$\hat{U}_{\epsilon_3} \cdot (\hat{U}_{\epsilon_2} \cdot \hat{U}_{\epsilon_1}) = (\hat{U}_{\epsilon_3} \cdot \hat{U}_{\epsilon_2}) \cdot \hat{U}_{\epsilon_1}. \quad (3.36)$$

13. Suppose that \hat{U}_ϵ is an arbitrary translation operator and $\phi(x)$ is an arbitrary function of space-time coordinate, then

$$\hat{U}_\epsilon \phi(x) \hat{U}_\epsilon^{-1} = f(x - \epsilon(x)). \quad (3.37)$$

This relation is quite useful in following discussions.

14. Suppose that \hat{U}_ϵ is an arbitrary translation operator. Define

$$\Lambda^\alpha{}_\beta = \frac{\partial x^\alpha}{\partial (x - \epsilon(x))^\beta}, \quad (3.38)$$

$$\Lambda_\alpha{}^\beta = \frac{\partial (x - \epsilon(x))^\beta}{\partial x^\alpha}. \quad (3.39)$$

They satisfy

$$\Lambda_\alpha{}^\mu \Lambda^\alpha{}_\nu = \delta_\nu^\mu, \quad (3.40)$$

$$\Lambda_\mu{}^\alpha \Lambda^\nu{}_\alpha = \delta_\mu^\nu. \quad (3.41)$$

Then we have following relations:

$$\hat{U}_\epsilon \hat{P}_\alpha \hat{U}_\epsilon^{-1} = \Lambda^\beta{}_\alpha \hat{P}_\beta, \quad (3.42)$$

$$\hat{U}_\epsilon dx^\alpha \hat{U}_\epsilon^{-1} = \Lambda_\beta{}^\alpha dx^\beta. \quad (3.43)$$

These give out the the transformation laws of \hat{P}_α and dx^α under local gravitational gauge transformations.

Gravitational gauge group (GGG) is a transformation group which consists of all non-singular translation operators \hat{U}_ϵ . We can easily see that gravitational gauge group is indeed a group, for

1. the product of two arbitrary non-singular translation operators is also a non-singular translation operator, which is also an element of the gravitational gauge group. So, the product of the group satisfies closure property which is expressed in eq(3.31);
2. the product of the gravitational gauge group also satisfies the associative law which is expressed in eq(3.36);
3. the gravitational gauge group has its unit element $\mathbf{1}$, it satisfies

$$\mathbf{1} \cdot \hat{U}_\epsilon = \hat{U}_\epsilon \cdot \mathbf{1} = \hat{U}_\epsilon; \quad (3.44)$$

4. every non-singular element \hat{U}_ϵ has its inverse element which is given by eqs(3.33) and (3.35).

According to gauge principle, the gravitational gauge group is the symmetry of gravitational interactions. The global invariance of gravitational gauge transformation will give out conserved charges which is just the ordinary energy-momentum; the requirement of local gravitational gauge invariance needs introducing gravitational gauge field, and gravitational interactions are completely determined by the local gravitational gauge invariance.

The generators of gravitational gauge group is just the energy-momentum operators \hat{P}_α . This is required by gauge principle. It can also be seen from the form of infinitesimal transformations. Suppose that ϵ is an infinitesimal quantity, then we have

$$\hat{U}_\epsilon \simeq 1 - i\epsilon^\alpha \hat{P}_\alpha. \quad (3.45)$$

Therefore,

$$i \frac{\partial \hat{U}_\epsilon}{\partial \epsilon^\alpha} \Big|_{\epsilon=0} \quad (3.46)$$

gives out generators \hat{P}_α of gravitational gauge group. It is known that generators of gravitational gauge group are commute each other

$$[\hat{P}_\alpha, \hat{P}_\beta] = 0. \quad (3.47)$$

However, the commutation property of generators does not mean that gravitational gauge group is an Abel group, because two general elements of gravitational gauge group do not commute:

$$[\hat{U}_{\epsilon_1}, \hat{U}_{\epsilon_2}] \neq 0. \quad (3.48)$$

Gravitational gauge group is a kind of non-Abel gauge group. The non-Abel nature of gravitational gauge group will cause self-interactions of gravitational gauge field.

In order to avoid confusion, we need to pay some attention to some differences between two concepts: space-time translation group and gravitational gauge group. Generally speaking, space-time translation is a kind of coordinates transformation, that is, the objects or fields in space-time are kept fixed while the space-time coordinates that describe the motion of objective matter undergo transformation. But gravitational gauge transformation is a kind of system transformation rather than a kind of coordinates transformation. In system transformation, the space-time coordinate system is kept unchanged while objects or fields undergo transformation. From mathematical point of view, space-time translation and gravitational gauge transformation are essentially the same, and the space-time translation is the inverse transformation of the gravitational gauge transformation; but from physical point of view, space-time translation and gravitational gauge transformation are quite different, especially when we discuss gravitational gauge transformation of gravitational

gauge field. For gravitational gauge field, its gravitational gauge transformation is not the inverse transformation of its space-time translation. In a meaning, space-time translation is a kind of mathematical transformation, which contains little dynamical information of interactions; while gravitational gauge transformation is a kind of physical transformation, which contains all dynamical information of interactions and is convenient for us to study physical interactions. Through gravitational gauge symmetry, we can determine the whole gravitational interactions among various kinds of fields. This is the reason why we do not call gravitational gauge transformation space-time translation. This is important for all of our discussions on gravitational gauge transformations of various kinds of fields.

Suppose that $\phi(x)$ is an arbitrary scalar field. Its gravitational gauge transformation is

$$\phi(x) \rightarrow \phi'(x) = (\hat{U}_\epsilon \phi(x)). \quad (3.49)$$

Similar to ordinary $SU(N)$ non-able gauge field theory, there are two kinds of scalars. For example, in chiral perturbative theory, the ordinary π mesons are scalar fields, but they are vector fields in isospin space. Similar case exists in gravitational gauge field theory. A Lorentz scalar can be a scalar or a vector or a tensor in the space of gravitational gauge group. If $\phi(x)$ is a scalar in the space of gravitational gauge group, we just simply denote it as $\phi(x)$ in gauge group space. If it is a vector in the space of gravitational gauge group, it can be expanded in the gravitational gauge group space in the following way:

$$\phi(x) = \phi^\alpha(x) \cdot \hat{P}_\alpha. \quad (3.50)$$

The transformation of component field is

$$\phi^\alpha(x) \rightarrow \phi'^\alpha(x) = \Lambda^\alpha_\beta \hat{U}_\epsilon \phi^\beta(x) \hat{U}_\epsilon^{-1}. \quad (3.51)$$

The important thing that we must remember is that, the α index is not a Lorentz index, it is just a group index. For gravitation gauge group, it is quite special that a group index looks like a Lorentz index. We must be carefully on this important thing. This will cause some fundamental changes on quantum gravity. Lorentz scalar $\phi(x)$ can also be a tensor in gauge group space. suppose that it is a n th order tensor in gauge group space, then it can be expanded as

$$\phi(x) = \phi^{\alpha_1 \dots \alpha_n}(x) \cdot \hat{P}_{\alpha_1} \dots \hat{P}_{\alpha_n}. \quad (3.52)$$

The transformation of component field is

$$\phi^{\alpha_1 \dots \alpha_n}(x) \rightarrow \phi'^{\alpha_1 \dots \alpha_n}(x) = \Lambda^{\alpha_1}_{\beta_1} \dots \Lambda^{\alpha_n}_{\beta_n} \hat{U}_\epsilon \phi^{\beta_1 \dots \beta_n}(x) \hat{U}_\epsilon^{-1}. \quad (3.53)$$

If $\phi(x)$ is a spinor field, the above discussion is also valid. That is, a spinor can also be a scalar or a vector or a tensor in the space of gravitational gauge group. The gravitational gauge transformations of the component fields are also given by eqs.(3.49-53). There is no transformations in spinor space, which is different from that of the Lorentz transformation of a spinor.

Suppose that $A_\mu(x)$ is an arbitrary vector field. Here, the index μ is a Lorentz index. Its gravitational gauge transformation is:

$$A_\mu(x) \rightarrow A'_\mu(x) = (\hat{U}_\epsilon A_\mu(x)). \quad (3.54)$$

Please remember that there is no rotation in the space of Lorentz index μ , while in the general coordinates transformations of general relativity, there is rotation in the space of Lorentz index μ . The reason is that gravitational gauge transformation is a kind of system transformation, while in general relativity, the general coordinates transformation is a kind of coordinates transformation. If $A_\mu(x)$ is a scalar in the space of gravitational gauge group, eq(3.54) is all for its gauge transformation. If $A_\mu(x)$ is a vector in the space of gravitational gauge group, it can be expanded as:

$$A_\mu(x) = A_\mu^\alpha(x) \cdot \hat{P}_\alpha. \quad (3.55)$$

The transformation of component field is

$$A_\mu^\alpha(x) \rightarrow A'^\alpha_\mu(x) = \Lambda^\alpha_\beta \hat{U}_\epsilon A_\mu^\beta(x) \hat{U}_\epsilon^{-1}. \quad (3.56)$$

If $A_\mu(x)$ is a n th order tensor in the space of gravitational gauge group, then

$$A_\mu(x) = A_\mu^{\alpha_1 \dots \alpha_n}(x) \cdot \hat{P}_{\alpha_1} \dots \hat{P}_{\alpha_n}. \quad (3.57)$$

The transformation of component fields is

$$A_\mu^{\alpha_1 \dots \alpha_n}(x) \rightarrow A'^{\alpha_1 \dots \alpha_n}_\mu(x) = \Lambda^{\alpha_1}_{\beta_1} \dots \Lambda^{\alpha_n}_{\beta_n} \hat{U}_\epsilon A_\mu^{\beta_1 \dots \beta_n}(x) \hat{U}_\epsilon^{-1}. \quad (3.58)$$

Therefore, under gravitational gauge transformations, the behavior of a group index is quite different from that of a Lorentz index. However, they have the same behavior in global Lorentz transformations.

Generally speaking, suppose that $T_{\nu_1 \dots \nu_m}^{\mu_1 \dots \mu_n}(x)$ is an arbitrary tensor, its gravitational gauge transformations are:

$$T_{\nu_1 \dots \nu_m}^{\mu_1 \dots \mu_n}(x) \rightarrow T'_{\nu_1 \dots \nu_m}^{\mu_1 \dots \mu_n}(x) = (\hat{U}_\epsilon T_{\nu_1 \dots \nu_m}^{\mu_1 \dots \mu_n}(x)). \quad (3.59)$$

If it is a p th order tensor in group space, then

$$T_{\nu_1 \dots \nu_m}^{\mu_1 \dots \mu_n}(x) = T_{\nu_1 \dots \nu_m}^{\mu_1 \dots \mu_n; \alpha_1 \dots \alpha_p}(x) \cdot \hat{P}_{\alpha_1} \dots \hat{P}_{\alpha_p}. \quad (3.60)$$

The transformation of component fields is

$$T_{\nu_1 \dots \nu_m}^{\mu_1 \dots \mu_n; \alpha_1 \dots \alpha_p}(x) \rightarrow T'_{\nu_1 \dots \nu_m}{}^{\mu_1 \dots \mu_n; \alpha_1 \dots \alpha_p}(x) = \Lambda^{\alpha_1}_{\beta_1} \dots \Lambda^{\alpha_p}_{\beta_p} \hat{U}_\epsilon T_{\nu_1 \dots \nu_m}^{\mu_1 \dots \mu_n; \beta_1 \dots \beta_p}(x) \hat{U}_\epsilon^{-1}. \quad (3.61)$$

Finally, we give definitions of three useful tensors. η_2 is a Lorentz scalar, but it is a second order tensor in group space. That is,

$$\eta_2 = \eta_2^{\alpha\beta} \cdot \hat{P}_\alpha \hat{P}_\beta. \quad (3.62)$$

The gravitational gauge transformation of η_2 is

$$\eta_2 \rightarrow \eta'_2 = (\hat{U}_\epsilon \eta_2). \quad (3.63)$$

The transformation of component field is

$$\eta_2^{\alpha\beta} \rightarrow \eta_2'^{\alpha\beta} = \Lambda^\alpha_{\alpha_1} \Lambda^\beta_{\beta_1} \hat{U}_\epsilon \eta_2^{\alpha_1 \beta_1} \hat{U}_\epsilon^{-1}. \quad (3.64)$$

As a symbolic operation, we use $\eta_2^{\alpha\beta}$ to raise a group index or to descend a group index. Covariant group metric tensor $\eta_{2\alpha\beta}$ is defined by

$$\eta_{2\alpha\beta} \eta_2^{\beta\gamma} = \eta_2^{\gamma\beta} \eta_{2\beta\alpha} = \delta_\alpha^\gamma \quad (3.65)$$

In a special representation of gravitational gauge group, $\eta_2^{\alpha\beta}$ is selected to be diagonal, that is:

$$\begin{aligned} \eta_2^{0\ 0} &= -1, \\ \eta_2^{1\ 1} &= 1, \\ \eta_2^{2\ 2} &= 1, \\ \eta_2^{3\ 3} &= 1, \end{aligned} \quad (3.66)$$

and other components of $\eta_2^{\alpha\beta}$ vanish. Group index of \hat{P}_α is raised by $\eta_2^{\alpha\beta}$. that is

$$\hat{P}^\alpha = \eta_2^{\alpha\beta} \hat{P}_\beta. \quad (3.67)$$

Eq.(3.66) is the simplest choice for $\eta_{2\alpha\beta}$. In this choice, the equation of motion of gravitational gauge field has the simplest expressions. Another choice for $\eta_2^{\alpha\beta}$ is given by the following relation[27],

$$\eta_2^{\alpha\beta} = g^{\alpha\beta} = \eta^{\mu\nu} (\delta_\mu^\alpha - g C_\mu^\alpha) (\delta_\nu^\beta - g C_\nu^\beta), \quad (3.68)$$

where $\eta^{\mu\nu}$ is the Minkowski metric, and C_μ^α is the gravitational gauge field which will be introduced in the next chapter. If we use eq.(3.68) to construct the lagrangian of gravitational system, the equation of motion of gravitational field will be much more complicated than that of choice eq.(3.66)[27]. Eq.(3.68) is a possible choice. Because

of the traditional belief that fundamental theory of fundamental interactions should be simple, we will choose the simplest choice which has the most beautiful form in this paper. In other words, in this paper, we select eq.(3.66) to be the definition of $\eta_2^{\alpha\beta}$. $\eta_2^{\alpha\beta}$ is not the metric in gravitational gauge group space when we use eq.(3.66) as definition, it is only a mathematical notation.

η_1^μ is a Lorentz vector. It is also a vector in the space of gravitational gauge group.

$$\eta_1^\mu = \eta_{1\alpha}^\mu \cdot \hat{P}^\alpha \quad (3.69)$$

For $\eta_{1\alpha}^\mu$, the index μ is a Lorentz index and the index α is a group index. The gravitational gauge transformation of η_1^μ is

$$\eta_1^\mu \rightarrow \eta_1^{\prime\mu} = (\hat{U}_\epsilon \eta_1^\mu). \quad (3.70)$$

The transformation of its component field is

$$\eta_{1\alpha}^\mu \rightarrow \eta_{1\alpha}^{\prime\mu} = \Lambda_\alpha^\beta \hat{U}_\epsilon \eta_{1\beta}^\mu \hat{U}_\epsilon^{-1}. \quad (3.71)$$

In a special coordinate system and a special representation of gravitational gauge group, $\eta_{1\alpha}^\mu$ is selected to be δ -function, that is

$$\eta_{1\alpha}^\mu = \delta_\alpha^\mu. \quad (3.72)$$

$\eta^{\mu\nu}$ is a second order Lorentz tensor, but it is a scalar in group space. It is the metric of the coordinate space. A Lorentz index can be raised or descended by this metric tensor. In a special coordinate system, it is selected to be:

$$\begin{aligned} \eta^{0\ 0} &= -1, \\ \eta^{1\ 1} &= 1, \\ \eta^{2\ 2} &= 1, \\ \eta^{3\ 3} &= 1, \end{aligned} \quad (3.73)$$

and other components of $\eta^{\mu\nu}$ vanish. $\eta^{\mu\nu}$ is the traditional Minkowski metric.

4 Pure Gravitational Gauge Fields

Before we study gravitational field, we must determine which field represents gravitational field. In the traditional gravitational gauge theory, gravitational field is represented by space-time metric tensor. If there is gravitational field in space-time,

the space-time metric will not be equivalent to Minkowski metric, and space-time will become curved. In other words, in the traditional gravitational gauge theory, quantum gravity is formulated in curved space-time. In this paper, we will not follow this way. The underlying point of view of this new quantum gauge theory of gravity is that gravitational field is represented by gauge potential, and curved space-time is considered to be a classical effect of macroscopic gravitational field. When we study quantum gravity, which mainly concerns microscopic gravitational interactions, we will not inherit the conception of curved space-time. In other words, if we put gravity into the structure of space-time, the space-time will become curved and there will be no physical gravity in space-time, because all gravitational effects are put into space-time metric and gravity is geometrized. But if we study physical gravitational interactions, it is better to rescue gravity from space-time metric and treat gravity as a physical field. In this case, space-time is flat and there is physical gravity in Minkowski space-time. For this reason, we will not introduce the concept of curved space-time to study quantum gravity in this paper. The space-time is always flat, gravitational field is represented by gauge potential and gravitational interactions are always treated as physical interactions. This point of view will be discussed again later. Besides, according to gauge principle, it is required that gravitational field is represented by gauge potential.

Now, let's begin to construct the Lagrangian of gravitational gauge theory. For the sake of simplicity, let's suppose that $\phi(x)$ is a Lorentz scalar and gauge group scalar. According to above discussions, Its gravitational gauge transformation is:

$$\phi(x) \rightarrow \phi'(x) = (\hat{U}_\epsilon \phi(x)). \quad (4.1)$$

Because

$$(\partial_\mu \hat{U}_\epsilon) \neq 0, \quad (4.2)$$

partial differential of $\phi(x)$ does not transform covariantly under gravitational gauge transformation:

$$\partial_\mu \phi(x) \rightarrow \partial_\mu \phi'(x) \neq (\hat{U}_\epsilon \partial_\mu \phi(x)). \quad (4.3)$$

In order to construct an action which is invariant under local gravitational gauge transformation, gravitational gauge covariant derivative is highly necessary. The gravitational gauge covariant derivative is defined by

$$D_\mu = \partial_\mu - igC_\mu(x), \quad (4.4)$$

where $C_\mu(x)$ is the gravitational gauge field. It is a Lorentz vector. Under gravitational gauge transformations, it transforms as

$$C_\mu(x) \rightarrow C'_\mu(x) = \hat{U}_\epsilon(x)C_\mu(x)\hat{U}_\epsilon^{-1}(x) + \frac{i}{g}\hat{U}_\epsilon(x)(\partial_\mu \hat{U}_\epsilon^{-1}(x)). \quad (4.5)$$

Using the original definition of \hat{U}_ϵ , we can strictly proved that

$$[\partial_\mu, \hat{U}_\epsilon] = (\partial_\mu \hat{U}_\epsilon). \quad (4.6)$$

Therefor, we have

$$\hat{U}_\epsilon \partial_\mu \hat{U}_\epsilon^{-1} = \partial_\mu + \hat{U}_\epsilon (\partial_\mu \hat{U}_\epsilon^{-1}), \quad (4.7)$$

$$\hat{U}_\epsilon D_\mu \hat{U}_\epsilon^{-1} = \partial_\mu - ig C'_\mu(x). \quad (4.8)$$

So, under local gravitational gauge transformations,

$$D_\mu \phi(x) \rightarrow D'_\mu \phi'(x) = (\hat{U}_\epsilon D_\mu \phi(x)), \quad (4.9)$$

$$D_\mu(x) \rightarrow D'_\mu(x) = \hat{U}_\epsilon D_\mu(x) \hat{U}_\epsilon^{-1}. \quad (4.10)$$

Gravitational gauge field $C_\mu(x)$ is vector field, it is a Lorentz vector. It is also a vector in gauge group space, so it can be expanded as linear combinations of generators of gravitational gauge group:

$$C_\mu(x) = C_\mu^\alpha(x) \cdot \hat{P}_\alpha. \quad (4.11)$$

C_μ^α are component fields of gravitational gauge field. They looks like a second rank tensor. But according to our previous discussion, they are not tensor fields, they are vector fields. The index α is not a Lorentz index, it is just a gauge group index. Gravitational gauge field C_μ^α has only one Lorentz index, so it is a kind of vector field. This is an inevitable results of gauge principle. The gravitational gauge transformation of component fields is:

$$C_\mu^\alpha(x) \rightarrow C'^\alpha_\mu(x) = \Lambda^\alpha_{\beta} (\hat{U}_\epsilon C_\mu^\beta(x)) - \frac{1}{g} (\hat{U}_\epsilon \partial_\mu \epsilon^\alpha(y)), \quad (4.12)$$

where y is a function of space-time coordinates which satisfy:

$$(\hat{U}_\epsilon y(x)) = x. \quad (4.13)$$

The strength of gravitational gauge field is defined by

$$F_{\mu\nu} = \frac{1}{-ig} [D_\mu, D_\nu], \quad (4.14)$$

or

$$F_{\mu\nu} = \partial_\mu C_\nu(x) - \partial_\nu C_\mu(x) - ig C_\mu(x) C_\nu(x) + ig C_\nu(x) C_\mu(x). \quad (4.15)$$

$F_{\mu\nu}$ is a second order Lorentz tensor. It is a vector in group space, so it can be expanded in group space,

$$F_{\mu\nu}(x) = F_{\mu\nu}^\alpha(x) \cdot \hat{P}_\alpha. \quad (4.16)$$

The explicit form of component field strengths is

$$F_{\mu\nu}^\alpha = \partial_\mu C_\nu^\alpha - \partial_\nu C_\mu^\alpha - g C_\mu^\beta \partial_\beta C_\nu^\alpha + g C_\nu^\beta \partial_\beta C_\mu^\alpha \quad (4.17)$$

The strength of gravitational gauge fields transforms covariantly under gravitational gauge transformation:

$$F_{\mu\nu} \rightarrow F'_{\mu\nu} = \hat{U}_\epsilon F_{\mu\nu} \hat{U}_\epsilon^{-1}. \quad (4.18)$$

The gravitational gauge transformation of the field strength of component field is

$$F_{\mu\nu}^\alpha \rightarrow F'^\alpha_{\mu\nu} = \Lambda^\alpha{}_\beta (\hat{U}_\epsilon F_{\mu\nu}^\beta). \quad (4.19)$$

Similar to traditional gauge field theory, the kinematical term for gravitational gauge field can be selected as

$$\mathcal{L}_0 = -\frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} \eta_{\alpha\beta} F_{\mu\nu}^\alpha F_{\rho\sigma}^\beta. \quad (4.20)$$

We can easily prove that this Lagrangian does not invariant under gravitational gauge transformation, it transforms covariantly

$$\mathcal{L}_0 \rightarrow \mathcal{L}'_0 = (\hat{U}_\epsilon \mathcal{L}_0). \quad (4.21)$$

In order to resume the gravitational gauge symmetry of the action, we introduce an extremely important factor, whose form is

$$J(C) = e^{I(C)} = e^{g\eta_{1\alpha}^\mu C_\mu^\alpha}, \quad (4.22)$$

where

$$I(C) = g\eta_{1\alpha}^\mu C_\mu^\alpha. \quad (4.23)$$

The choice of $J(C)$ is not unique[27]. another choice of $J(C)$ is

$$J(C) = \sqrt{-\det g_{\alpha\beta}}, \quad (4.22a)$$

where $g_{\alpha\beta}$ is given by eq.(9.30). Under gravitational gauge transformations, $g_{\alpha\beta}$ transforms as

$$g_{\alpha\beta} \rightarrow g'_{\alpha\beta} = \Lambda_\alpha{}^{\alpha_1} \Lambda_\beta{}^{\beta_1} g_{\alpha_1\beta_1}.$$

Then $J(C)$ transforms as

$$J(C) \rightarrow J'(C') = J \cdot (\hat{U}_\epsilon J(C)),$$

where J is the Jacobian of the transformation. Eq.(4.22) gives out the simplest and most natural choice for $J(C)$, and in this case, the equation of motion of gravitational gauge field has simplest and most explicit expressions. Because of the traditional belief that fundamental theory of fundamental interactions should be simple, we will choose the simplest choice which has the most beautiful form in this paper. On the other hand, the choice given by eq.(4.22) can make the whole theory renormalizable, therefore we use eq.(4.22) as the definition of $J(C)$ in this paper. In fact, $J(C)$ times any gravitational gauge covariant functions of pure gravitational gauge field C_μ^α can be regarded as a possible choice of $J(C)$. Some detailed discussions on the second choice can be found in literature [27]. When we use eq.(4.22) as the definition of $J(C)$, then, the Lagrangian for gravitational gauge field is selected as

$$\mathcal{L} = J(C)\mathcal{L}_0 = e^{I(C)}\mathcal{L}_0, \quad (4.24)$$

and the action for gravitational gauge field is

$$S = \int d^4x \mathcal{L}. \quad (4.25)$$

It can be proved that this action has gravitational gauge symmetry. In other words, it is invariant under gravitational gauge transformation,

$$S \rightarrow S' = S. \quad (4.26)$$

In order to prove the gravitational gauge symmetry of the action, the following relation is important,

$$\int d^4x J(\hat{U}_\epsilon f(x)) = \int d^4x f(x), \quad (4.27)$$

where $f(x)$ is an arbitrary function of space-time coordinate and J is the Jacobian of the corresponding transformation,

$$J = \det\left(\frac{\partial(x - \epsilon)^\mu}{\partial x^\nu}\right). \quad (4.28)$$

According to gauge principle, the global gauge symmetry will give out conserved charges. Now, let's discuss the conserved charges of global gravitational gauge transformation. Suppose that ϵ^α is an infinitesimal constant 4-vector. Then, in the first order approximation, we have

$$\hat{U}_\epsilon = 1 - \epsilon^\alpha \partial_\alpha + o(\epsilon^2). \quad (4.29)$$

The first order variation of the gravitational gauge field is

$$\delta C_\mu^\alpha(x) = -\epsilon^\nu \partial_\nu C_\mu^\alpha, \quad (4.30)$$

and the first order variation of action is:

$$\delta S = \int d^4x \epsilon^\alpha \partial_\mu T_{i\alpha}^\mu, \quad (4.31)$$

where $T_{i\alpha}^\mu$ is the inertial energy-momentum tensor, whose definition is

$$T_{i\alpha}^\mu \equiv e^{I(C)} \left(-\frac{\partial \mathcal{L}_0}{\partial \partial_\mu C_\nu^\beta} \partial_\alpha C_\nu^\beta + \delta_\alpha^\mu \mathcal{L}_0 \right). \quad (4.32)$$

It is a conserved current,

$$\partial_\mu T_{i\alpha}^\mu = 0. \quad (4.33)$$

Except for the factor $e^{I(C)}$, the form of the inertial energy-momentum tensor is almost completely the same as that in the traditional quantum field theory. It means that gravitational interactions will change energy-momentum of matter fields, which is what we expected in general relativity.

The Euler-Lagrange equation for gravitational gauge field is

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu C_\nu^\alpha} = \frac{\partial \mathcal{L}}{\partial C_\nu^\alpha}. \quad (4.34)$$

This form is completely that same as what we have ever seen in quantum field theory. But if we insert eq.(4.24) into it, we will get

$$\partial_\mu \frac{\partial \mathcal{L}_0}{\partial \partial_\mu C_\nu^\alpha} = \frac{\partial \mathcal{L}_0}{\partial C_\nu^\alpha} + g \eta_{1\alpha}^\nu \mathcal{L}_0 - g \partial_\mu (\eta_{1\gamma}^\rho C_\rho^\gamma) \frac{\partial \mathcal{L}_0}{\partial \partial_\mu C_\nu^\alpha}. \quad (4.35)$$

Eq.(4.17) can be changed into

$$F_{\mu\nu}^\alpha = (D_\mu C_\nu^\alpha) - (D_\nu C_\mu^\alpha), \quad (4.36)$$

so the Lagrangian \mathcal{L}_0 depends on gravitational gauge fields completely through its covariant derivative. Therefore,

$$\frac{\partial \mathcal{L}_0}{\partial C_\nu^\alpha} = -g \frac{\partial \mathcal{L}_0}{\partial D_\nu C_\mu^\beta} \partial_\alpha C_\mu^\beta. \quad (4.37)$$

Using the above relations and

$$\frac{\partial \mathcal{L}_0}{\partial \partial_\mu C_\nu^\alpha} = -\eta^{\mu\lambda} \eta^{\nu\tau} \eta_{2\alpha\beta} F_{\lambda\tau}^\beta - g \eta^{\nu\lambda} \eta^{\sigma\tau} \eta_{2\alpha\beta} F_{\lambda\tau}^\beta C_\sigma^\mu, \quad (4.38)$$

the above equation of motion of gravitational gauge fields are changed into:

$$\begin{aligned}\partial_\mu(\eta^{\mu\lambda}\eta^{\nu\tau}\eta_{2\alpha\beta}F_{\lambda\tau}^\beta) &= -g\left(-\frac{\partial\mathcal{L}_0}{\partial D_\nu C_\mu^\beta}\partial_\alpha C_\mu^\beta + \eta_{1\alpha}^\nu\mathcal{L}_0\right) \\ &\quad + g\partial_\mu(\eta_{1\gamma}^\rho C_\rho^\gamma)\frac{\partial\mathcal{L}_0}{\partial\partial_\mu C_\nu^\alpha} \\ &\quad - g\partial_\mu(\eta^{\nu\lambda}\eta^{\sigma\tau}\eta_{2\alpha\beta}F_{\lambda\tau}^\beta C_\sigma^\mu).\end{aligned}\tag{4.39}$$

If we define

$$\begin{aligned}T_{g\alpha}^\nu &= -\frac{\partial\mathcal{L}_0}{\partial D_\nu C_\mu^\beta}\partial_\alpha C_\mu^\beta + \eta_{1\alpha}^\nu\mathcal{L}_0 \\ &\quad - \partial_\mu(\eta_{1\gamma}^\rho C_\rho^\gamma)\frac{\partial\mathcal{L}_0}{\partial\partial_\mu C_\nu^\alpha} \\ &\quad + \partial_\mu(\eta^{\nu\lambda}\eta^{\sigma\tau}\eta_{2\alpha\beta}F_{\lambda\tau}^\beta C_\sigma^\mu),\end{aligned}\tag{4.40}$$

Then we can get a simpler form of equation of motion,

$$\partial_\mu(\eta^{\mu\lambda}\eta^{\nu\tau}\eta_{2\alpha\beta}F_{\lambda\tau}^\beta) = -gT_{g\alpha}^\nu.\tag{4.41}$$

$T_{g\alpha}^\nu$ is also a conserved current, that is

$$\partial_\nu T_{g\alpha}^\nu = 0,\tag{4.42}$$

because

$$\partial_\nu\partial_\mu(\eta^{\mu\lambda}\eta^{\nu\tau}\eta_{2\alpha\beta}F_{\lambda\tau}^\beta) = 0.\tag{4.43}$$

$T_{g\alpha}^\nu$ is called gravitational energy-momentum tensor, which is the source of gravitational gauge field. Now we get two different energy-momentum tensors, one is the inertial energy-momentum tensor $T_{i\alpha}^\nu$ and another is the gravitational energy-momentum tensor $T_{g\alpha}^\nu$. They are similar, but they are different. The inertial energy-momentum tensor $T_{i\alpha}^\nu$ is given by conservation law which is associate with global gravitational gauge symmetry, it gives out an energy-momentum 4-vector:

$$P_{i\alpha} = \int d^3\vec{x} T_{i\alpha}^0.\tag{4.44}$$

It is a conserved charges,

$$\frac{d}{dt}P_{i\alpha} = 0.\tag{4.45}$$

The time component of $P_{i\alpha}$, that is P_{i0} , gives out the Hamiltonian H of the system,

$$H = -P_{i0} = \int d^3\vec{x} e^{I(C)}(\pi_\alpha^\mu \dot{C}_\mu^\alpha - \mathcal{L}_0).\tag{4.46}$$

According to our conventional belief, H should be the inertial energy of the system, therefore $P_{i\alpha}$ is the inertial energy-momentum of the system. The gravitational energy-momentum is given by equation of motion of gravitational gauge field, it is also a conserved current. The space integration of the time component of it gives out a conserved energy-momentum 4-vector,

$$P_{g\alpha} = \int d^3\vec{x} T_{g\alpha}^0.\tag{4.47}$$

It is also a conserved charge,

$$\frac{d}{dt}P_{g\alpha} = 0. \quad (4.48)$$

The time component of it just gives out the gravitational energy of the system. This can be easily seen. Set ν and α in eq.(4.41) to 0, we get

$$\partial^i F_{i0}^0 = -gT_{g0}^0. \quad (4.49)$$

The field strength of gravitational field is defined by

$$E^i = -F_{i0}^0. \quad (4.50)$$

The space integration of eq.(4.49) gives out

$$\oint d\vec{\sigma} \cdot \vec{E} = g \int d^3\vec{x} T_{g0}^0. \quad (4.51)$$

According to Newton's classical theory of gravity, $\int d^3\vec{x} T_{g0}^0$ in the right hand term is just the gravitational mass of the system. Denote the gravitational mass of the system as M_g , that is

$$M_g = - \int d^3\vec{x} T_{g0}^0. \quad (4.52)$$

Then eq(4.51) is changed into

$$\oint d\vec{\sigma} \cdot \vec{E} = -gM_g. \quad (4.53)$$

This is just the classical Newton's law of universal gravitation. It can be strictly proved that gravitational mass is different from inertial mass. They are not equivalent. But their difference is at least second order infinitesimal quantity if the gravitational field C_μ^α and the gravitational coupling constant g are all first order infinitesimal quantities, for this difference is proportional to gC_μ^α . So, this difference is too small to be detected in experiments. But in the environment of strong gravitational field, the difference will become relatively larger and will be easier to be detected. Much more highly precise measurement of this difference is strongly needed to test this prediction and to test the validity of the equivalence principle. In the chapter of classical limit of quantum gauge theory of gravity, we will return to discuss this problem again.

Now, let's discuss self-coupling of gravitational field. The Lagrangian of gravitational gauge field is given by eq(4.24). Because

$$e^{g\eta_{1\alpha}^\mu C_\mu^\alpha} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} (g\eta_{1\alpha}^\mu C_\mu^\alpha)^n, \quad (4.54)$$

there are vertexes of n gravitational gauge fields in tree diagram with n can be arbitrary integer number that is greater than 3. This property is important for renormalization of the theory. Because the coupling constant of the gravitational gauge interactions has negative mass dimension, any kind of regular vertex exists divergence. In order to cancel these divergences, we need to introduce the corresponding counterterms. Because of the existence of the vertex of n gravitational gauge fields in tree diagram in the non-renormalized Lagrangian, we need not introduce any new counterterm which does not exist in the non-renormalized Lagrangian, what we need to do is to redefine gravitational coupling constant g and gravitational gauge field C_μ^α in renormalization. If there is no $e^{I(C)}$ term in the original Lagrangian, then we will have to introduce infinite counterterms in renormalization, and therefore the theory is non-renormalizable. Because of the existence of the factor $e^{I(C)}$, though quantum gauge theory of gravity looks like a non-renormalizable theory according to power counting law, it is indeed renormalizable. In a word, the factor $e^{I(C)}$ is highly important for the quantum gauge theory of gravity.

5 Gravitational Interactions of Scalar Fields

Now, let's start to discuss gravitational interactions of matter fields. First, we discuss gravitational interactions of scalar fields. For the sake of simplicity, we first discuss real scalar field. Suppose that $\phi(x)$ is a real scalar field. The traditional Lagrangian for the real scalar field is

$$-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x) - \frac{m^2}{2}\phi^2(x), \quad (5.1)$$

where m is the mass of scalar field. This is the Lagrangian for a free real scalar field. The Euler-Lagrangian equation of motion of it is

$$(\eta^{\mu\nu}\partial_\mu\partial_\nu - m^2)\phi(x) = 0, \quad (5.2)$$

which is the famous Klein-Gordan equation.

Now, replace the ordinary partial derivative ∂_μ with gauge covariant derivative D_μ , and add the Lagrangian of pure gravitational gauge field, we get

$$\mathcal{L}_0 = -\frac{1}{2}\eta^{\mu\nu}(D_\mu\phi)(D_\nu\phi) - \frac{m^2}{2}\phi^2 - \frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}\eta_{2\alpha\beta}F_{\mu\nu}^\alpha F_{\rho\sigma}^\beta. \quad (5.3)$$

The full Lagrangian is selected to be

$$\mathcal{L} = e^{I(C)}\mathcal{L}_0, \quad (5.4)$$

and the action S is defined by

$$S = \int d^4x \mathcal{L}. \quad (5.5)$$

Using our previous definitions of gauge covariant derivative D_μ and strength of gravitational gauge field $F_{\mu\nu}^\alpha$, we can obtain an explicit form of Lagrangian \mathcal{L} ,

$$\mathcal{L} = \mathcal{L}_F + \mathcal{L}_I, \quad (5.6)$$

with \mathcal{L}_F the free Lagrangian and \mathcal{L}_I the interaction Lagrangian. Their explicit expressions are

$$\begin{aligned} \mathcal{L}_F &= -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x) - \frac{m^2}{2}\phi^2(x) - \frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}\eta_{2\alpha\beta}F_{0\mu\nu}^\alpha F_{0\rho\sigma}^\beta, \quad (5.7) \\ \mathcal{L}_I &= \mathcal{L}_F \cdot (\sum_{n=1}^{\infty} \frac{1}{n!} (g\eta_{1\alpha_1}^{\mu_1} C_{\mu_1}^{\alpha_1})^n) \\ &\quad + g e^{I(C)} \eta^{\mu\nu} C_\mu^\alpha (\partial_\alpha \phi) (\partial_\nu \phi) - \frac{g^2}{2} e^{I(C)} \eta^{\mu\nu} C_\mu^\alpha C_\nu^\beta (\partial_\alpha \phi) (\partial_\beta \phi) \\ &\quad + g e^{I(C)} \eta^{\mu\rho} \eta^{\nu\sigma} \eta_{2\alpha\beta} (\partial_\mu C_\nu^\alpha - \partial_\nu C_\mu^\alpha) C_\rho^\delta \partial_\delta C_\sigma^\beta \\ &\quad - \frac{1}{2} g^2 e^{I(C)} \eta^{\mu\rho} \eta^{\nu\sigma} \eta_{2\alpha\beta} (C_\mu^\delta \partial_\delta C_\nu^\alpha - C_\nu^\delta \partial_\delta C_\mu^\alpha) C_\rho^\epsilon \partial_\epsilon C_\sigma^\beta, \end{aligned} \quad (5.8)$$

where,

$$F_{0\mu\nu}^\alpha = \partial_\mu C_\nu^\alpha - \partial_\nu C_\mu^\alpha. \quad (5.9)$$

From eq.(5.8), we can see that scalar field can directly couples to any number of gravitational gauge fields. This is one of the most important interaction properties of gravity. Other kinds of interactions, such as strong interactions, weak interactions and electromagnetic interactions do not have this kind of interaction properties. Because the gravitational coupling constant has negative mass dimension, renormalization of theory needs this kind of interaction properties. In other words, if matter field can not directly couple to any number of gravitational gauge fields, the theory will be non-renormalizable.

The symmetries of the theory can be easily seen from eq.(5.3). First, let's discuss Lorentz symmetry. In eq.(5.3), some indexes are Lorentz indexes and some are group indexes. Lorentz indexes and group indexes have different transformation law under gravitational gauge transformation, but they have the same transformation law under Lorentz transformation. Therefore, it can be easily seen that both \mathcal{L}_0 and $I(C)$ are Lorentz scalars, the Lagrangian \mathcal{L} and action S are invariant under global Lorentz transformation.

Under gravitational gauge transformations, real scalar field $\phi(x)$ transforms as

$$\phi(x) \rightarrow \phi'(x) = (\hat{U}_\epsilon \phi(x)), \quad (5.10)$$

therefore,

$$D_\mu \phi(x) \rightarrow D'_\mu \phi'(x) = (\hat{U}_\epsilon D_\mu \phi(x)). \quad (5.11)$$

It can be easily proved that \mathcal{L}_0 transforms covariantly

$$\mathcal{L}_0 \rightarrow \mathcal{L}'_0 = (\hat{U}_\epsilon \mathcal{L}_0), \quad (5.12)$$

and the action eq.(5.5) of the system is invariant,

$$S \rightarrow S' = S. \quad (5.13)$$

Please remember that eq.(4.27) is an important relation to be used in the proof of the gravitational gauge symmetry of the action.

Global gravitational gauge symmetry gives out conserved charges. Suppose that \hat{U}_ϵ is an infinitesimal gravitational gauge transformation, it will have the form of eq.(4.29). The first order variations of fields are

$$\delta C_\mu^\alpha(x) = -\epsilon^\nu (\partial_\nu C_\mu^\alpha(x)), \quad (5.14)$$

$$\delta \phi(x) = -\epsilon^\nu (\partial_\nu \phi(x)), \quad (5.15)$$

Using Euler-Lagrange equation of motions for scalar fields and gravitational gauge fields, we can obtain that

$$\delta S = \int d^4x \epsilon^\alpha \partial_\mu T_{i\alpha}^\mu, \quad (5.16)$$

where

$$T_{i\alpha}^\mu \equiv e^{I(C)} \left(-\frac{\partial \mathcal{L}_0}{\partial \partial_\mu \phi} \partial_\alpha \phi - \frac{\partial \mathcal{L}_0}{\partial \partial_\mu C_\nu^\beta} \partial_\alpha C_\nu^\beta + \delta_\alpha^\mu \mathcal{L}_0 \right). \quad (5.17)$$

Because action is invariant under global gravitational gauge transformation,

$$\delta S = 0, \quad (5.18)$$

and ϵ^α is an arbitrary infinitesimal constant 4-vector, we obtain,

$$\partial_\mu T_{i\alpha}^\mu = 0. \quad (5.19)$$

This is the conservation equation for inertial energy-momentum tensor. $T_{i\alpha}^\mu$ is the conserved current which corresponds to the global gravitational gauge symmetry. The space integration of the time component of inertial energy-momentum tensor gives out the conserved charge, which is just the inertial energy-momentum of the

system. The time component of the conserved charge is the Hamilton of the system, which is

$$H = -P_{i=0} = \int d^3 \vec{x} e^{I(C)} (\pi_\phi \dot{\phi} + \pi_\alpha^\mu \dot{C}_\mu^\alpha - \mathcal{L}_0), \quad (5.20)$$

where π_ϕ is the canonical conjugate momentum of the real scalar field. The inertial space momentum of the system is given by

$$P^i = P_{i=i} = \int d^3 \vec{x} e^{I(C)} (-\pi_\phi \partial_i \phi - \pi_\alpha^\mu \partial_i C_\mu^\alpha). \quad (5.21)$$

According to gauge principle, after quantization, they will become generators of quantum gravitational gauge transformation.

There is an important and interesting relation that can be easily obtained from the Lagrangian which is given by eq.(5.3). Define

$$G_\mu^\alpha = \delta_\mu^\alpha - g C_\mu^\alpha. \quad (5.22)$$

Then Lagrangian given by eq(5.3) can be changed into

$$\mathcal{L}_0 = -\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - \frac{m^2}{2} \phi^2 - \frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} \eta_{2\alpha\beta} F_{\mu\nu}^\alpha F_{\rho\sigma}^\beta, \quad (5.23)$$

where

$$g^{\alpha\beta} \equiv \eta^{\mu\nu} G_\mu^\alpha G_\nu^\beta. \quad (5.24)$$

$g^{\alpha\beta}$ is the metric tensor of curved space-time in the classical limit of the quantum gauge theory of gravity. We can easily see that, when there is no gravitational field in space-time, that is,

$$C_\mu^\alpha = 0, \quad (5.25)$$

the classical space-time will be flat

$$g^{\alpha\beta} = \eta^{\alpha\beta}. \quad (5.26)$$

This is what we expected in general relativity. We do not talk to much on this problem here, for we will discuss this problem again in details in the chapter on the classical limit of the quantum gauge theory of gravity.

Euler-Lagrange equations of motion can be easily deduced from action principle. Keep gravitational gauge field C_μ^α fixed and let real scalar field vary infinitesimally, then the first order infinitesimal variation of action is

$$\delta S = \int d^4 x e^{I(C)} \left(\frac{\partial \mathcal{L}_0}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}_0}{\partial \partial_\mu \phi} - g \partial_\mu (\eta_{1\alpha}^\nu C_\nu^\alpha) \frac{\partial \mathcal{L}_0}{\partial \partial_\mu \phi} \right) \delta \phi. \quad (5.27)$$

Because $\delta\phi$ is an arbitrary variation of scalar field, according to action principle, we get

$$\frac{\partial\mathcal{L}_0}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}_0}{\partial\partial_\mu\phi} - g\partial_\mu(\eta_{1\alpha}^\nu C_\nu^\alpha) \frac{\partial\mathcal{L}_0}{\partial\partial_\mu\phi} = 0. \quad (5.28)$$

Because of the existence of the factor $e^{I(C)}$, the equation of motion for scalar field is quite different from the traditional form in quantum field theory. But the difference is a second order infinitesimal quantity if we suppose that both gravitational coupling constant and gravitational gauge field are first order infinitesimal quantities. Because

$$\frac{\partial\mathcal{L}_0}{\partial\partial_\alpha\phi} = -g^{\alpha\beta}\partial_\beta\phi, \quad (5.29)$$

$$\frac{\partial\mathcal{L}_0}{\partial\phi} = -m^2\phi, \quad (5.30)$$

the explicit form of the equation of motion of scalar field is

$$g^{\alpha\beta}\partial_\alpha\partial_\beta\phi - m^2\phi + (\partial_\alpha g^{\alpha\beta})\partial_\beta\phi + gg^{\alpha\beta}(\partial_\beta\phi)\partial_\alpha(\eta_{1\gamma}^\nu C_\nu^\gamma) = 0. \quad (5.31)$$

The equation of motion for gravitational gauge field is:

$$\partial_\mu(\eta^{\mu\lambda}\eta^{\nu\tau}\eta_{2\alpha\beta}F_{\lambda\tau}^\beta) = -gT_{g\alpha}^\nu, \quad (5.32)$$

where $T_{g\alpha}^\nu$ is the gravitational energy-momentum tensor, whose definition is:

$$\begin{aligned} T_{g\alpha}^\nu &= -\frac{\partial\mathcal{L}_0}{\partial D_\nu C_\mu^\beta}\partial_\alpha C_\mu^\beta - \frac{\partial\mathcal{L}_0}{\partial D_\nu\phi}\partial_\alpha\phi + \eta_{1\alpha}^\nu\mathcal{L}_0 \\ &\quad -\partial_\mu(\eta_{1\gamma}^\rho C_\rho^\gamma)\frac{\partial\mathcal{L}_0}{\partial\partial_\mu C_\nu^\alpha} + \partial_\mu(\eta^{\nu\lambda}\eta^{\sigma\tau}\eta_{2\alpha\beta}F_{\lambda\tau}^\beta C_\sigma^\mu), \end{aligned} \quad (5.33)$$

We can see again that, for matter field, its inertial energy-momentum tensor is also different from the gravitational energy-momentum tensor, this difference completely originate from the influences of gravitational gauge field. Compare eq.(5.33) with eq.(5.17), and set gravitational gauge field to zero, that is

$$D_\mu\phi = \partial_\mu\phi, \quad (5.34)$$

$$I(C) = 0, \quad (5.35)$$

then we find that two energy-momentum tensors are completely the same:

$$T_{i\alpha}^\mu = T_{g\alpha}^\mu. \quad (5.36)$$

It means that the equivalence principle only strictly hold in a space-time where there is no gravitational field. In the environment of strong gravitational field, such as in

black hole, the equivalence principle will be strongly violated.

Define

$$\mathbf{L} = \int d^3 \vec{x} \mathcal{L} = \int d^3 \vec{x} e^{I(C)} \mathcal{L}_0. \quad (5.37)$$

Then, we can easily prove that

$$\frac{\delta \mathbf{L}}{\delta \phi} = e^{I(C)} \left(\frac{\partial \mathcal{L}_0}{\partial \phi} - \partial_i \frac{\partial \mathcal{L}_0}{\partial \partial_i \phi} - g \partial_i (\eta_{1\alpha}^\mu C_\mu^\alpha) \frac{\partial \mathcal{L}_0}{\partial \partial_i \phi} \right), \quad (5.38)$$

$$\frac{\delta \mathbf{L}}{\delta \dot{\phi}} = e^{I(C)} \frac{\partial \mathcal{L}_0}{\partial \dot{\phi}}, \quad (5.39)$$

$$\frac{\delta \mathbf{L}}{\delta C_\nu^\alpha} = e^{I(C)} \left(\frac{\partial \mathcal{L}_0}{\partial C_\nu^\alpha} - \partial_i \frac{\partial \mathcal{L}_0}{\partial \partial_i C_\nu^\alpha} + g \eta_{1\alpha}^\nu \mathcal{L}_0 - g \partial_i (\eta_{1\beta}^\mu C_\mu^\beta) \frac{\partial \mathcal{L}_0}{\partial \partial_i C_\nu^\alpha} \right), \quad (5.40)$$

$$\frac{\delta \mathbf{L}}{\delta \dot{C}_\nu^\alpha} = e^{I(C)} \frac{\partial \mathcal{L}_0}{\partial \dot{C}_\nu^\alpha}. \quad (5.41)$$

Then, Hamilton's action principle gives out the following equations of motion:

$$\frac{\delta \mathbf{L}}{\delta \phi} - \frac{d}{dt} \frac{\delta \mathbf{L}}{\delta \dot{\phi}} = 0, \quad (5.42)$$

$$\frac{\delta \mathbf{L}}{\delta C_\nu^\alpha} - \frac{d}{dt} \frac{\delta \mathbf{L}}{\delta \dot{C}_\nu^\alpha} = 0. \quad (5.43)$$

These two equations of motion are essentially the same as the Euler-Lagrange equations of motion which we have obtained before. But these two equations have more beautiful forms.

The Hamiltonian of the system is given by a Legendre transformation,

$$\begin{aligned} H &= \int d^3 \vec{x} \left(\frac{\delta \mathbf{L}}{\delta \dot{\phi}} \dot{\phi} + \frac{\delta \mathbf{L}}{\delta \dot{C}_\mu^\alpha} \dot{C}_\mu^\alpha \right) - \mathbf{L} \\ &= \int d^3 \vec{x} e^{I(C)} (\pi_\phi \dot{\phi} + \pi_\alpha^\mu \dot{C}_\mu^\alpha - \mathcal{L}_0), \end{aligned} \quad (5.44)$$

where,

$$\pi_\phi = \frac{\partial \mathcal{L}_0}{\partial \dot{\phi}}, \quad (5.45)$$

$$\pi_\alpha^\mu = \frac{\partial \mathcal{L}_0}{\partial \dot{C}_\mu^\alpha}. \quad (5.46)$$

It can be easily seen that the Hamiltonian given by Legendre transformation is completely the same as that given by inertial energy-momentum tensor. After Legendre transformation, ϕ , C_μ^α , $e^{I(C)}\pi_\phi$ and $e^{I(C)}\pi_\alpha^\mu$ are canonical independent variables. Let these variables vary infinitesimally, we can get

$$\frac{\delta H}{\delta \phi} = -\frac{\delta \mathbf{L}}{\delta \phi}, \quad (5.47)$$

$$\frac{\delta H}{\delta(e^{I(C)}\pi_\phi)} = \dot{\phi}, \quad (5.48)$$

$$\frac{\delta H}{\delta C_\nu^\alpha} = -\frac{\delta \mathbf{L}}{\delta C_\nu^\alpha}, \quad (5.49)$$

$$\frac{\delta H}{\delta(e^{I(C)}\pi_\alpha^\nu)} = \dot{C}_\nu^\alpha. \quad (5.50)$$

Then, Hamilton's equations of motion read:

$$\frac{d}{dt}\phi = \frac{\delta H}{\delta(e^{I(C)}\pi_\phi)}, \quad (5.51)$$

$$\frac{d}{dt}(e^{I(C)}\pi_\phi) = -\frac{\delta H}{\delta \phi}, \quad (5.52)$$

$$\frac{d}{dt}C_\nu^\alpha = \frac{\delta H}{\delta(e^{I(C)}\pi_\alpha^\nu)}, \quad (5.53)$$

$$\frac{d}{dt}(e^{I(C)}\pi_\alpha^\nu) = -\frac{\delta H}{\delta C_\nu^\alpha}. \quad (5.54)$$

The forms of the Hamilton's equations of motion are completely the same as those appears in usual quantum field theory and usual classical analytical mechanics. Therefor, the introduction of the factor $e^{I(C)}$ does not affect the forms of Lagrange equations of motion and Hamilton's equations of motion.

The Poisson brackets of two general functional of canonical arguments can be defined by

$$\begin{aligned} \{A, B\} = & \int d^3 \vec{x} \left(\frac{\delta A}{\delta \phi} \frac{\delta B}{\delta(e^{I(C)}\pi_\phi)} - \frac{\delta A}{\delta(e^{I(C)}\pi_\phi)} \frac{\delta B}{\delta \phi} \right. \\ & \left. + \frac{\delta A}{\delta C_\nu^\alpha} \frac{\delta B}{\delta(e^{I(C)}\pi_\alpha^\nu)} - \frac{\delta A}{\delta(e^{I(C)}\pi_\alpha^\nu)} \frac{\delta B}{\delta C_\nu^\alpha} \right). \end{aligned} \quad (5.55)$$

According to this definition, we have

$$\{\phi(\vec{x}, t), (e^{I(C)}\pi_\phi)(\vec{y}, t)\} = \delta^3(\vec{x} - \vec{y}), \quad (5.56)$$

$$\{C_\nu^\alpha(\vec{x}, t), (e^{I(C)}\pi_\beta^\mu)(\vec{y}, t)\} = \delta_\nu^\mu \delta_\beta^\alpha \delta^3(\vec{x} - \vec{y}). \quad (5.57)$$

These two relations can be used as the starting point of canonical quantization of quantum gravity.

Using Poisson brackets, the Hamilton's equations of motion can be expressed in another forms,

$$\frac{d}{dt}\phi(\vec{x}, t) = \{\phi(\vec{x}, t) , H\}, \quad (5.58)$$

$$\frac{d}{dt}(e^{I(C)}\pi_\phi)(\vec{x}, t) = \{(e^{I(C)}\pi_\phi)(\vec{x}, t) , H\}, \quad (5.59)$$

$$\frac{d}{dt}C_\nu^\alpha(\vec{x}, t) = \{C_\nu^\alpha(\vec{x}, t) , H\}, \quad (5.60)$$

$$\frac{d}{dt}(e^{I(C)}\pi_\alpha^\nu)(\vec{x}, t) = \{(e^{I(C)}\pi_\alpha^\nu)(\vec{x}, t) , H\}. \quad (5.61)$$

Therefore, if A is an arbitrary functional of the canonical arguments ϕ , C_μ^α , $e^{I(C)}\pi_\phi$ and $e^{I(C)}\pi_\alpha^\mu$, then we have

$$\dot{A} = \{A , H\}. \quad (5.62)$$

After quantization, this equation will become the Heisenberg equation.

If $\phi(x)$ is a complex scalar field, its traditional Lagrangian is

$$-\eta^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi^*(x) - m^2\phi(x)\phi^*(x). \quad (5.63)$$

Replace ordinary partial derivative with gauge covariant derivative, and add into the Lagrangian for pure gravitational gauge field, we get,

$$\mathcal{L}_0 = -\eta^{\mu\nu}D_\mu\phi(D_\nu\phi)^* - m^2\phi\phi^* - \frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}\eta_{2\alpha\beta}F_{\mu\nu}^\alpha F_{\rho\sigma}^\beta. \quad (5.64)$$

Repeating all above discussions, we can get the whole theory for gravitational interactions of complex scalar fields. We will not repeat this discussion here.

6 Gravitational Interactions of Dirac Field

In the usual quantum field theory, the Lagrangian for Dirac field is

$$-\bar{\psi}(\gamma^\mu\partial_\mu + m)\psi. \quad (6.1)$$

Replace ordinary partial derivative with gauge covariant derivative, and add into the Lagrangian of pure gravitational gauge field, we get,

$$\mathcal{L}_0 = -\bar{\psi}(\gamma^\mu D_\mu + m)\psi - \frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}\eta_{2\alpha\beta}F_{\mu\nu}^\alpha F_{\rho\sigma}^\beta. \quad (6.2)$$

The full Lagrangian of the system is

$$\mathcal{L} = e^{I(C)} \mathcal{L}_0, \quad (6.3)$$

and the corresponding action is

$$S = \int d^4x \mathcal{L} = \int d^4x e^{I(C)} \mathcal{L}_0. \quad (6.4)$$

This Lagrangian can be separated into two parts,

$$\mathcal{L} = \mathcal{L}_F + \mathcal{L}_I, \quad (6.5)$$

with \mathcal{L}_F the free Lagrangian and \mathcal{L}_I the interaction Lagrangian. Their explicit forms are

$$\mathcal{L}_F = -\bar{\psi}(\gamma^\mu \partial_\mu + m)\psi - \frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}\eta_{2\alpha\beta}F_{0\mu\nu}^\alpha F_{0\rho\sigma}^\beta, \quad (6.6)$$

$$\begin{aligned} \mathcal{L}_I &= \mathcal{L}_F \cdot (\sum_{n=1}^{\infty} \frac{1}{n!} (g\eta_{1\alpha_1}^{\mu_1} C_{\mu_1}^{\alpha_1})^n) + ge^{I(C)}\bar{\psi}\gamma^\mu(\partial_\alpha\psi)C_\mu^\alpha \\ &+ ge^{I(C)}\eta^{\mu\rho}\eta^{\nu\sigma}\eta_{2\alpha\beta}(\partial_\mu C_\nu^\alpha - \partial_\nu C_\mu^\alpha)C_\rho^\delta\partial_\delta C_\sigma^\beta \\ &- \frac{1}{2}g^2e^{I(C)}\eta^{\mu\rho}\eta^{\nu\sigma}\eta_{2\alpha\beta}(C_\mu^\delta\partial_\delta C_\nu^\alpha - C_\nu^\delta\partial_\delta C_\mu^\alpha)C_\rho^\epsilon\partial_\epsilon C_\sigma^\beta. \end{aligned} \quad (6.7)$$

From \mathcal{L}_I , we can see that Dirac field can directly couple to any number of gravitational gauge fields, the mass term of Dirac field also take part in gravitational interactions. All these interactions are completely determined by the requirement of gravitational gauge symmetry. The Lagrangian function before renormalization almost contains all kind of divergent vertex, which is important in the renormalization of the theory. Besides, from eq.(6.7), we can directly write out Feynman rules of the corresponding interaction vertexes.

Because the traditional Lagrangian function eq.(6.1) is invariant under global Lorentz transformation, which is already proved in the traditional quantum field theory, and the covariant derivative has the same behavior as partial derivative under global Lorentz transformation, the first two terms of Lagrangian \mathcal{L} are global Lorentz invariant. We have already prove that the Lagrangian function for pure gravitational gauge field is invariant under global Lorentz transformation. Therefor, \mathcal{L} has global Lorentz symmetry.

The gravitational gauge transformation of Dirac field is

$$\psi(x) \rightarrow \psi'(x) = (\hat{U}_\epsilon\psi(x)). \quad (6.8)$$

$\bar{\psi}$ transforms similarly,

$$\bar{\psi}(x) \rightarrow \bar{\psi}'(x) = (\hat{U}_\epsilon\bar{\psi}(x)). \quad (6.9)$$

Dirac γ -matrices is not a physical field, so it keeps unchanged under gravitational gauge transformation,

$$\gamma^\mu \rightarrow \gamma^\mu. \quad (6.10)$$

It can be proved that, under gravitational gauge transformation, \mathcal{L}_0 transforms as

$$\mathcal{L}_0 \rightarrow \mathcal{L}'_0 = (\hat{U}_\epsilon \mathcal{L}_0). \quad (6.11)$$

So,

$$\mathcal{L} \rightarrow \mathcal{L}' = J(\hat{U}_\epsilon \mathcal{L}_0), \quad (6.12)$$

where J is the Jacobi of the corresponding space-time translation. Then using eq.(4.27), we can prove that the action S has gravitational gauge symmetry.

Suppose that \hat{U}_ϵ is an infinitesimal global transformation, then the first order infinitesimal variations of Dirac field are

$$\delta\psi = -\epsilon^\nu \partial_\nu \psi, \quad (6.13)$$

$$\delta\bar{\psi} = -\epsilon^\nu \partial_\nu \bar{\psi}. \quad (6.14)$$

The first order variation of action is

$$\delta S = \int d^4x \epsilon^\alpha \partial_\mu T_{i\alpha}^\mu, \quad (6.15)$$

where $T_{i\alpha}^\mu$ is the inertial energy-momentum tensor whose definition is,

$$T_{i\alpha}^\mu \equiv e^{I(C)} \left(-\frac{\partial \mathcal{L}_0}{\partial \partial_\mu \psi} \partial_\alpha \psi - \frac{\partial \mathcal{L}_0}{\partial \partial_\mu C_\nu^\beta} \partial_\alpha C_\nu^\beta + \delta_\alpha^\mu \mathcal{L}_0 \right). \quad (6.16)$$

The global gravitational gauge symmetry of action gives out conservation equation of the inertial energy-momentum tensor,

$$\partial_\mu T_{i\alpha}^\mu = 0. \quad (6.17)$$

The inertial energy-momentum tensor is the conserved current which expected by gauge principle. The space integration of its time component gives out the conserved energy-momentum of the system,

$$H = -P_{i0} = \int d^3 \vec{x} e^{I(C)} (\pi_\psi \dot{\psi} + \pi_\alpha^\mu \dot{C}_\mu^\alpha - \mathcal{L}_0), \quad (6.18)$$

$$P^i = P_{i\ i} = \int d^3 \vec{x} e^{I(C)} (-\pi_\psi \partial_i \psi - \pi_\alpha^\mu \partial_i C_\mu^\alpha), \quad (6.19)$$

where

$$\pi_\psi = \frac{\partial \mathcal{L}_0}{\partial \dot{\psi}}. \quad (6.20)$$

The equation of motion for Dirac field is

$$(\gamma^\mu D_\mu + m)\psi = 0. \quad (6.21)$$

From this expression, we can see that the factor $e^{I(C)}$ does not affect the equation of motion of Dirac field. This is caused by the asymmetric form of the Lagrangian. If we use a symmetric form of Lagrangian, the factor $e^{I(C)}$ will also affect the equation of motion of Dirac field, which will be discussed later.

The equation of motion for gravitational fields can be easily deduced,

$$\partial_\mu(\eta^{\mu\lambda}\eta^{\nu\tau}\eta_{2\alpha\beta}F_{\lambda\tau}^\beta) = -gT_{g\alpha}^\nu, \quad (6.22)$$

where $T_{g\alpha}^\nu$ is the gravitational energy-momentum tensor, whose definition is:

$$\begin{aligned} T_{g\alpha}^\nu &= -\frac{\partial\mathcal{L}_0}{\partial D_\nu C_\mu^\beta}\partial_\alpha C_\mu^\beta - \frac{\partial\mathcal{L}_0}{\partial D_\nu\psi}\partial_\alpha\psi + \eta_{1\alpha}^\nu\mathcal{L}_0 \\ &\quad -\partial_\mu(\eta_{1\gamma}^\rho C_\rho^\gamma)\frac{\partial\mathcal{L}_0}{\partial\partial_\mu C_\nu^\sigma} + \partial_\mu(\eta^{\nu\lambda}\eta^{\sigma\tau}\eta_{2\alpha\beta}F_{\lambda\tau}^\beta C_\sigma^\mu). \end{aligned} \quad (6.23)$$

We see again that the gravitational energy-momentum tensor is different from the inertial energy-momentum tensor.

In usual quantum field theory, the Lagrangian for Dirac field has a more symmetric form, which is

$$-\bar{\psi}(\gamma^\mu \overleftrightarrow{\partial}_\mu + m)\psi, \quad (6.24)$$

where

$$\overleftrightarrow{\partial}_\mu = \frac{\partial_\mu - \overleftarrow{\partial}_\mu}{2}. \quad (6.25)$$

The Euler-Lagrange equation of motion of eq.(6.24) also gives out the conventional Dirac equation.

Now replace ordinary space-time partial derivative with covariant derivative, and add into the Lagrangian of pure gravitational gauge field, we get,

$$\mathcal{L}_0 = -\bar{\psi}(\gamma^\mu \overleftrightarrow{D}_\mu + m)\psi - \frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}\eta_{2\alpha\beta}F_{\mu\nu}^\alpha F_{\rho\sigma}^\beta, \quad (6.26)$$

where $\overleftrightarrow{D}_\mu$ is defined by

$$\overleftrightarrow{D}_\mu = \frac{D_\mu - \overleftarrow{D}_\mu}{2}. \quad (6.27)$$

Operator \overleftarrow{D}_μ is understood in the following way

$$f(x) \overleftarrow{D}_\mu g(x) = (D_\mu f(x))g(x), \quad (6.28)$$

with $f(x)$ and $g(x)$ two arbitrary functions. The Lagrangian density \mathcal{L} and action S are also defined by eqs.(6.3-4). In this case, the free Lagrangian \mathcal{L}_F and interaction Lagrangian \mathcal{L}_I are given by

$$\mathcal{L}_F = -\bar{\psi}(\gamma^\mu \overleftrightarrow{\partial}_\mu + m)\psi - \frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}\eta_{2\alpha\beta}F_{0\mu\nu}^\alpha F_{0\rho\sigma}^\beta, \quad (6.29)$$

$$\begin{aligned} \mathcal{L}_I &= \mathcal{L}_F \cdot (\sum_{n=1}^{\infty} \frac{1}{n!} (g\eta_{1\alpha_1}^{\mu_1} C_{\mu_1}^{\alpha_1})^n) + ge^{I(C)}(\bar{\psi}\gamma^\mu \overleftrightarrow{\partial}_\alpha \psi)C_\mu^\alpha \\ &+ ge^{I(C)}\eta^{\mu\rho}\eta^{\nu\sigma}\eta_{2\alpha\beta}(\partial_\mu C_\nu^\alpha - \partial_\nu C_\mu^\alpha)C_\rho^\delta \partial_\delta C_\sigma^\beta \\ &- \frac{1}{2}g^2 e^{I(C)}\eta^{\mu\rho}\eta^{\nu\sigma}\eta_{2\alpha\beta}(C_\mu^\delta \partial_\delta C_\nu^\alpha - C_\nu^\delta \partial_\delta C_\mu^\alpha)C_\rho^\epsilon \partial_\epsilon C_\sigma^\beta. \end{aligned} \quad (6.30)$$

The Euler-Lagrange equation of motion for Dirac field is

$$\frac{\partial \mathcal{L}_0}{\partial \bar{\psi}} - \partial_\mu \frac{\partial \mathcal{L}_0}{\partial \partial_\mu \bar{\psi}} - g\partial_\mu (\eta_{1\alpha}^\nu C_\nu^\alpha) \frac{\partial \mathcal{L}_0}{\partial \partial_\mu \bar{\psi}} = 0. \quad (6.31)$$

Because

$$\frac{\partial \mathcal{L}_0}{\partial \bar{\psi}} = e^{I(C)}(-\frac{1}{2}\gamma^\mu D_\mu \psi - m\psi), \quad (6.32)$$

$$\frac{\partial \mathcal{L}_0}{\partial \partial_\mu \psi} = e^{I(C)}(\frac{1}{2}\gamma^\alpha G_\alpha^\mu \psi), \quad (6.33)$$

eq.(6.31) will be changed into

$$(\gamma^\mu D_\mu + m)\psi = -\frac{1}{2}\gamma^\mu (\partial_\alpha G_\mu^\alpha)\psi - g\gamma^\mu \psi D_\mu (\eta_{1\beta}^\nu C_\nu^\beta). \quad (6.34)$$

If gravitational gauge field vanishes, this equation of motion will return to the traditional Dirac equation.

The inertial energy-momentum tensor now becomes

$$T_{i\alpha}^\mu = e^{I(C)}(-\frac{\partial \mathcal{L}_0}{\partial \partial_\mu \psi} \partial_\alpha \psi - (\partial_\alpha \bar{\psi}) \frac{\partial \mathcal{L}_0}{\partial \partial_\mu \bar{\psi}} - \frac{\partial \mathcal{L}_0}{\partial \partial_\mu C_\nu^\beta} \partial_\alpha C_\nu^\beta + \delta_\alpha^\mu \mathcal{L}_0), \quad (6.35)$$

and the gravitational energy-momentum tensor becomes

$$\begin{aligned} T_{g\alpha}^\nu &= -\frac{\partial \mathcal{L}_0}{\partial D_\nu C_\mu^\beta} \partial_\alpha C_\mu^\beta - \frac{\partial \mathcal{L}_0}{\partial D_\nu \psi} \partial_\alpha \psi - (\partial_\alpha \bar{\psi}) \frac{\partial \mathcal{L}_0}{\partial D_\nu \bar{\psi}} + \eta_{1\alpha}^\nu \mathcal{L}_0 \\ &- \partial_\mu (\eta_{1\gamma}^\rho C_\rho^\gamma) \frac{\partial \mathcal{L}_0}{\partial \partial_\mu C_\nu^\alpha} + \partial_\mu (\eta^{\nu\lambda} \eta^{\sigma\tau} \eta_{2\alpha\beta} F_{\lambda\tau}^\beta C_\sigma^\alpha). \end{aligned} \quad (6.36)$$

Both of them are conserved energy-momentum tensor. But they are not equivalent.

7 Gravitational Interactions of Vector Field

The traditional Lagrangian for vector field is

$$-\frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}A_{\mu\nu}A_{\rho\sigma} - \frac{m^2}{2}\eta^{\mu\nu}A_\mu A_\nu, \quad (7.1)$$

where $A_{\mu\nu}$ is the strength of vector field which is given by

$$\partial_\mu A_\nu - \partial_\nu A_\mu. \quad (7.2)$$

The Lagrangian \mathcal{L}_0 that describes gravitational interactions between vector field and gravitational fields is

$$\mathcal{L}_0 = -\frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}A_{\mu\nu}A_{\rho\sigma} - \frac{m^2}{2}\eta^{\mu\nu}A_\mu A_\nu - \frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}\eta_{2\alpha\beta}F_{\mu\nu}^\alpha F_{\rho\sigma}^\beta. \quad (7.3)$$

In eq.(7.3), the definition of strength $A_{\mu\nu}$ is not given by eq.(7.2), it is given by

$$\begin{aligned} A_{\mu\nu} &= D_\mu A_\nu - D_\nu A_\mu \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu - gC_\mu^\alpha \partial_\alpha A_\nu + gC_\nu^\alpha \partial_\alpha A_\mu, \end{aligned} \quad (7.4)$$

where D_μ is the gravitational gauge covariant derivative, whose definition is given by eq.(4.4). The full Lagrangian \mathcal{L} is given by,

$$\mathcal{L} = e^{I(C)} \mathcal{L}_0. \quad (7.5)$$

The action S is defined by

$$S = \int d^4x \mathcal{L}. \quad (7.6)$$

The Lagrangian \mathcal{L} can be separated into two parts: the free Lagrangian \mathcal{L}_F and interaction Lagrangian \mathcal{L}_I . The explicit forms of them are

$$\mathcal{L}_F = -\frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}A_{0\mu\nu}A_{0\rho\sigma} - \frac{m^2}{2}\eta^{\mu\nu}A_\mu A_\nu - \frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}\eta_{2\alpha\beta}F_{0\mu\nu}^\alpha F_{0\rho\sigma}^\beta, \quad (7.7)$$

$$\begin{aligned} \mathcal{L}_I &= \mathcal{L}_F \cdot (\sum_{n=1}^{\infty} \frac{1}{n!} (g\eta_{1\alpha_1}^{\mu_1} C_{\mu_1}^{\alpha_1})^n) + ge^{I(C)}\eta^{\mu\rho}\eta^{\nu\sigma}A_{0\mu\nu}C_\rho^\alpha \partial_\alpha A_\sigma \\ &\quad - \frac{g^2}{2}e^{I(C)}\eta^{\mu\rho}\eta^{\nu\sigma}(C_\mu^\alpha C_\rho^\beta (\partial_\alpha A_\nu)(\partial_\beta A_\sigma) - C_\nu^\alpha C_\rho^\beta (\partial_\alpha A_\mu)(\partial_\beta A_\sigma)) \\ &\quad + ge^{I(C)}\eta^{\mu\rho}\eta^{\nu\sigma}\eta_{2\alpha\beta}(\partial_\mu C_\nu^\alpha - \partial_\nu C_\mu^\alpha)C_\rho^\delta \partial_\delta C_\sigma^\beta \\ &\quad - \frac{1}{2}g^2e^{I(C)}\eta^{\mu\rho}\eta^{\nu\sigma}\eta_{2\alpha\beta}(C_\mu^\delta \partial_\delta C_\nu^\alpha - C_\nu^\delta \partial_\delta C_\mu^\alpha)C_\rho^\epsilon \partial_\epsilon C_\sigma^\beta, \end{aligned} \quad (7.8)$$

where $A_{0\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The first two lines of \mathcal{L}_I contain interactions between vector field and gravitational gauge fields. It can be seen that the vector field can also directly couple to arbitrary number of gravitational gauge fields, which is one of the most important properties of gravitational gauge interactions. This interaction properties are required and determined by local gravitational gauge symmetry.

Under Lorentz transformations, group index and Lorentz index have the same behavior. Therefore every term in the Lagrangian \mathcal{L} are Lorentz scalar, and the whole Lagrangian \mathcal{L} and action S have Lorentz symmetry.

Under gravitational gauge transformations, vector field A_μ transforms as

$$A_\mu(x) \rightarrow A'_\mu(x) = (\hat{U}_\epsilon A_\mu(x)). \quad (7.9)$$

$D_\mu A_\nu$ and $A_{\mu\nu}$ transform covariantly,

$$D_\mu A_\nu \rightarrow D'_\mu A'_\nu = (\hat{U}_\epsilon D_\mu A_\nu), \quad (7.10)$$

$$A_{\mu\nu} \rightarrow A'_{\mu\nu} = (\hat{U}_\epsilon A_{\mu\nu}). \quad (7.11)$$

So, the gravitational gauge transformations of \mathcal{L}_0 and \mathcal{L} respectively are

$$\mathcal{L}_0 \rightarrow \mathcal{L}'_0 = (\hat{U}_\epsilon \mathcal{L}_0), \quad (7.12)$$

$$\mathcal{L} \rightarrow \mathcal{L}' = J(\hat{U}_\epsilon \mathcal{L}_0). \quad (7.13)$$

The action of the system is gravitational gauge invariant.

The global gravitational gauge transformation gives out conserved current of gravitational gauge symmetry. Under infinitesimal global gravitational gauge transformation, the vector field A_μ transforms as

$$\delta A_\mu = -\epsilon^\alpha \partial_\alpha A_\mu. \quad (7.14)$$

The first order variation of action is

$$\delta S = \int d^4x \epsilon^\alpha \partial_\mu T_{i\alpha}^\mu, \quad (7.15)$$

where $T_{i\alpha}^\mu$ is the inertial energy-momentum tensor whose definition is,

$$T_{i\alpha}^\mu = e^{I(C)} \left(-\frac{\partial \mathcal{L}_0}{\partial \partial_\mu A_\nu} \partial_\alpha A_\nu - \frac{\partial \mathcal{L}_0}{\partial \partial_\mu C_\nu^\beta} \partial_\alpha C_\nu^\beta + \delta_\alpha^\mu \mathcal{L}_0 \right). \quad (7.16)$$

$T_{i\alpha}^\mu$ is a conserved current. The space integration of its time component gives out inertial energy-momentum of the system,

$$H = -P_{i\ 0} = \int d^3 \vec{x} e^{I(C)} (\pi^\mu \dot{A}_\mu + \pi_\alpha^\mu \dot{C}_\mu^\alpha - \mathcal{L}_0), \quad (7.17)$$

$$P^i = P_{i\ i} = \int d^3 \vec{x} e^{I(C)} (-\pi^\mu \partial_i A_\mu - \pi_\alpha^\mu \partial_i C_\mu^\alpha), \quad (7.18)$$

where

$$\pi^\mu = \frac{\partial \mathcal{L}_0}{\partial \dot{A}_\mu}. \quad (7.19)$$

The equation of motion for vector field is

$$\frac{\partial \mathcal{L}_0}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}_0}{\partial \partial_\mu A_\nu} - g \partial_\mu (\eta_{1\alpha}^\lambda C_\lambda^\alpha) \frac{\partial \mathcal{L}_0}{\partial \partial_\mu A_\nu} = 0. \quad (7.20)$$

From eq.(7.3), we can obtain

$$\frac{\partial \mathcal{L}_0}{\partial \partial_\mu A_\nu} = -\eta^{\lambda\rho} \eta^{\nu\sigma} G_\lambda^\mu A_{\rho\sigma}, \quad (7.21)$$

$$\frac{\partial \mathcal{L}_0}{\partial A_\nu} = -m^2 \eta^{\lambda\nu} A_\lambda. \quad (7.22)$$

Then, eq.(7.20) is changed into

$$\eta^{\mu\rho} \eta^{\nu\sigma} D_\mu A_{\rho\sigma} - m^2 \eta^{\mu\nu} A_\mu = -\eta^{\lambda\rho} \eta^{\nu\sigma} (\partial_\mu G_\lambda^\mu) A_{\rho\sigma} - g \eta^{\mu\rho} \eta^{\nu\sigma} A_{\rho\sigma} D_\mu (\eta_{1\alpha}^\mu C_\mu^\alpha). \quad (7.23)$$

The equation of motion of gravitational gauge field is

$$\partial_\mu (\eta^{\mu\lambda} \eta^{\nu\tau} \eta_{2\alpha\beta} F_{\lambda\tau}^\beta) = -g T_{g\alpha}^\nu, \quad (7.24)$$

where $T_{g\alpha}^\nu$ is the gravitational energy-momentum tensor,

$$\begin{aligned} T_{g\alpha}^\nu &= -\frac{\partial \mathcal{L}_0}{\partial D_\nu C_\mu^\beta} \partial_\alpha C_\mu^\beta - \frac{\partial \mathcal{L}_0}{\partial D_\nu A_\mu} \partial_\alpha A_\mu + \eta_{1\alpha}^\nu \mathcal{L}_0 \\ &\quad - \partial_\mu (\eta_{1\gamma}^\rho C_\rho^\gamma) \frac{\partial \mathcal{L}_0}{\partial \partial_\mu C_\nu^\alpha} + \partial_\mu (\eta^{\nu\lambda} \eta^{\sigma\tau} \eta_{2\alpha\beta} F_{\lambda\tau}^\beta C_\sigma^\mu). \end{aligned} \quad (7.25)$$

$T_{g\alpha}^\nu$ is also a conserved current. The space integration of its time component gives out the gravitational energy-momentum which is the source of gravitational interactions. It can be also seen that inertial energy-momentum tensor and gravitational energy-momentum tensor are not equivalent.

8 Gravitational Interactions of Gauge Fields

It is known that QED, QCD and unified electroweak theory are all gauge theories. In this chapter, we will discuss how to unify these gauge theories with gravitational gauge theory, and how to unify four different kinds of fundamental interactions formally.

First, let's discuss QED theory. As an example, let's discuss electromagnetic interactions of Dirac field. The traditional electromagnetic interactions between Dirac field ψ and electromagnetic field A_μ is

$$-\frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}A_{\mu\nu}A_{\rho\sigma} - \bar{\psi}(\gamma^\mu(\partial_\mu - ieA_\mu) + m)\psi. \quad (8.1)$$

The Lagrangian that describes gravitational gauge interactions between gravitational gauge field and Dirac field or electromagnetic field and describes electromagnetic interactions between Dirac field and electromagnetic field is

$$\mathcal{L}_0 = -\bar{\psi}(\gamma^\mu(D_\mu - ieA_\mu) + m)\psi - \frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}\mathbf{A}_{\mu\nu}\mathbf{A}_{\rho\sigma} - \frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}\eta_{2\alpha\beta}F_{\mu\nu}^\alpha F_{\rho\sigma}^\beta, \quad (8.2)$$

where D_μ is the gravitational gauge covariant derivative which is given by eq.(4.4) and the strength of electromagnetic field A_μ is

$$\mathbf{A}_{\mu\nu} = A_{\mu\nu} + gG_\alpha^{-1\lambda}A_\lambda F_{\mu\nu}^\alpha, \quad (8.3)$$

where $A_{\mu\nu}$ is given by eq.(7.4) and G^{-1} is given by eq.(9.24). The full Lagrangian density and the action of the system are respectively given by,

$$\mathcal{L} = e^{I(C)}\mathcal{L}_0, \quad (8.4)$$

$$S = \int d^4x \mathcal{L}. \quad (8.5)$$

The system given by above Lagrangian has both $U(1)$ gauge symmetry and gravitational gauge symmetry. Under $U(1)$ gauge transformations,

$$\psi(x) \rightarrow \psi'(x) = e^{-i\alpha(x)}\psi(x), \quad (8.6)$$

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) - \frac{1}{e}D_\mu\alpha(x), \quad (8.7)$$

$$C_\mu^\alpha(x) \rightarrow C_\mu'^\alpha(x) = C_\mu^\alpha(x). \quad (8.8)$$

It can be proved that the Lagrangian \mathcal{L} is invariant under $U(1)$ gauge transformation. Under gravitational gauge transformations,

$$\psi(x) \rightarrow \psi'(x) = (\hat{U}_\epsilon \psi(x)), \quad (8.9)$$

$$A_\mu(x) \rightarrow A'_\mu(x) = (\hat{U}_\epsilon A_\mu(x)), \quad (8.10)$$

$$C_\mu(x) \rightarrow C'_\mu(x) = \hat{U}_\epsilon(x) C_\mu(x) \hat{U}_\epsilon^{-1}(x) + \frac{i}{g} \hat{U}_\epsilon(x) (\partial_\mu \hat{U}_\epsilon^{-1}(x)). \quad (8.11)$$

The action S given by eq.(8.4) is invariant under gravitational gauge transformation.

Lagrangian \mathcal{L} can be separated into free Lagrangian \mathcal{L}_F and interaction Lagrangian \mathcal{L}_I ,

$$\mathcal{L} = \mathcal{L}_F + \mathcal{L}_I, \quad (8.12)$$

where

$$\mathcal{L}_F = -\frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} A_{0\mu\nu} A_{0\rho\sigma} - \bar{\psi} (\gamma^\mu \partial_\mu + m) \psi - \frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} \eta_{2\alpha\beta} F_{0\mu\nu}^\alpha F_{0\rho\sigma}^\beta, \quad (8.13)$$

$$\begin{aligned} \mathcal{L}_I = & \mathcal{L}_F \cdot (\sum_{n=1}^{\infty} \frac{1}{n!} (g \eta_{1\alpha_1}^{\mu_1} C_{\mu_1}^{\alpha_1})^n) + ie \cdot e^{I(C)} \bar{\psi} \gamma^\mu \psi A_\mu \\ & + g e^{I(C)} \bar{\psi} \gamma^\mu \partial_\alpha \psi C_\mu^\alpha + g e^{I(C)} \eta^{\mu\rho} \eta^{\nu\sigma} A_{0\mu\nu} C_\rho^\alpha \partial_\alpha A_\sigma \\ & - \frac{g}{2} e^{I(C)} \eta^{\mu\rho} \eta^{\nu\sigma} A_{\mu\nu} G_\alpha^{-1\lambda} A_\lambda F_{\rho\sigma}^\alpha \\ & - \frac{g^2}{4} e^{I(C)} \eta^{\mu\rho} \eta^{\nu\sigma} G_\alpha^{-1\kappa} G_\beta^{-1\lambda} A_\kappa A_\lambda F_{\mu\nu}^\alpha F_{\rho\sigma}^\beta \\ & - \frac{g^2}{2} e^{I(C)} \eta^{\mu\rho} \eta^{\nu\sigma} (C_\mu^\alpha C_\rho^\beta (\partial_\alpha A_\nu) (\partial_\beta A_\sigma) - C_\nu^\alpha C_\rho^\beta (\partial_\alpha A_\mu) (\partial_\beta A_\sigma)) \\ & + g e^{I(C)} \eta^{\mu\rho} \eta^{\nu\sigma} \eta_{2\alpha\beta} (\partial_\mu C_\nu^\alpha - \partial_\nu C_\mu^\alpha) C_\rho^\delta \partial_\delta C_\sigma^\beta \\ & - \frac{1}{2} g^2 e^{I(C)} \eta^{\mu\rho} \eta^{\nu\sigma} \eta_{2\alpha\beta} (C_\mu^\delta \partial_\delta C_\nu^\alpha - C_\nu^\delta \partial_\delta C_\mu^\alpha) C_\rho^\epsilon \partial_\epsilon C_\sigma^\beta. \end{aligned} \quad (8.14)$$

The traditional Lagrangian for QCD is

$$-\sum_n \bar{\psi}_n [\gamma^\mu (\partial_\mu - ig_c A_\mu^i \frac{\lambda_i}{2}) + m_n] \psi_n - \frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} A_{\mu\nu}^i A_{\rho\sigma}^i, \quad (8.15)$$

where ψ_n is the quark color triplet of the n th flavor, $A_{\mu\alpha}$ is the color gauge vector potential, $A_{\alpha\mu\nu}$ is the color gauge covariant field strength tensor, g_c is the strong

coupling constant, λ_α is the Gell-Mann matrix and m_n is the quark mass of the n th flavor. In gravitational gauge theory, this Lagrangian should be changed into

$$\begin{aligned} \mathcal{L}_0 = & -\sum_n \bar{\psi}_n [\gamma^\mu (D_\mu - ig_c A_\mu^i \frac{\lambda^i}{2}) + m_n] \psi_n - \frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} \mathbf{A}_{\mu\nu}^i \mathbf{A}_{\rho\sigma}^i \\ & - \frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} \eta_{2\alpha\beta} F_{\mu\nu}^\alpha F_{\rho\sigma}^\beta, \end{aligned} \quad (8.16)$$

where

$$\mathbf{A}_{\mu\nu}^i = A_{\mu\nu}^i + g G_\sigma^{-1\lambda} A_\lambda^i F_{\mu\nu}^\sigma, \quad (8.17)$$

$$A_{\mu\nu}^i = D_\mu A_\nu^i - D_\nu A_\mu^i + g_c f_{ijk} A_\mu^j A_\nu^k. \quad (8.18)$$

It can be proved that this system has both $SU(3)_c$ gauge symmetry and gravitational gauge symmetry. The unified electroweak model can be discussed in similar way.

Now, let's try to construct a theory which can describe all kinds of fundamental interactions in Nature. First we know that the fundamental particles that we know are fundamental fermions (such as leptons and quarks), gauge bosons (such as photon, gluons, gravitons and intermediate gauge bosons W^\pm and Z^0), and possible Higgs bosons. According to the Standard Model, leptons form left-hand doublets and right-hand singlets. Let's denote

$$\psi_L^{(1)} = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \quad \psi_L^{(2)} = \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \quad \psi_L^{(3)} = \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L, \quad (8.19)$$

$$\psi_R^{(1)} = e_R, \quad \psi_R^{(2)} = \mu_R, \quad \psi_R^{(3)} = \tau_R. \quad (8.20)$$

Neutrinos have no right-hand singlets. The weak hypercharge for left-hand doublets $\psi_L^{(i)}$ is -1 and for right-hand singlet $\psi_R^{(i)}$ is -2 . All leptons carry no color charge. In order to define the wave function for quarks, we have to introduce Kobayashi-Maskawa mixing matrix first, whose general form is,

$$K = \begin{pmatrix} c_1 & s_1 c_3 & s_1 s_3 \\ -s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} & c_1 c_2 s_3 + s_2 c_3 e^{i\delta} \\ s_1 s_2 & -c_1 s_2 c_3 - c_2 s_3 e^{i\delta} & -c_1 s_2 s_3 + c_2 c_3 e^{i\delta} \end{pmatrix} \quad (8.21)$$

where

$$c_i = \cos\theta_i, \quad s_i = \sin\theta_i \quad (i = 1, 2, 3) \quad (8.22)$$

and θ_i are generalized Cabibbo angles. The mixing between three different quarks d , s and b is given by

$$\begin{pmatrix} d_\theta \\ s_\theta \\ b_\theta \end{pmatrix} = K \begin{pmatrix} d \\ s \\ b \end{pmatrix}. \quad (8.23)$$

Quarks also form left-hand doublets and right-hand singlets,

$$q_L^{(1)a} = \begin{pmatrix} u_L^a \\ d_{\theta L}^a \end{pmatrix}, \quad q_L^{(2)a} = \begin{pmatrix} c_L^a \\ s_{\theta L}^a \end{pmatrix}, \quad q_L^{(3)a} = \begin{pmatrix} t_L^a \\ b_{\theta L}^a \end{pmatrix}, \quad (8.24)$$

$$\begin{aligned} q_u^{(1)a} &= u_R^a & q_u^{(2)a} &= c_R^a & q_u^{(3)a} &= t_R^a \\ q_{\theta d}^{(1)a} &= d_{\theta R}^a & q_{\theta d}^{(2)a} &= s_{\theta R}^a & q_{\theta d}^{(3)a} &= b_{\theta R}^a, \end{aligned} \quad (8.25)$$

where index a is color index. It is known that left-hand doublets have weak isospin $\frac{1}{2}$ and weak hypercharge $\frac{1}{3}$, right-hand singlets have no weak isospin, $q_u^{(j)a}$ s have weak hypercharge $\frac{4}{3}$ and $q_{\theta d}^{(j)a}$ s have weak hypercharge $-\frac{2}{3}$.

For gauge bosons, gravitational gauge field is also denoted by C_μ^α . The gluon field is denoted A_μ ,

$$A_\mu = A_\mu^i \frac{\lambda^i}{2}. \quad (8.26)$$

The color gauge covariant field strength tensor is also given by eq.(8.18). The $U(1)_Y$ gauge field is denoted by B_μ and $SU(2)$ gauge field is denoted by F_μ

$$F_\mu = F_\mu^n \frac{\sigma_n}{2}, \quad (8.27)$$

where σ_n is the Pauli matrix. The $U(1)_Y$ gauge field strength tensor is given by

$$\mathbf{B}_{\mu\nu} = B_{\mu\nu} + gG_\alpha^{-1\lambda} B_\lambda F_{\mu\nu}^\alpha, \quad (8.28a)$$

where

$$B_{\mu\nu} = D_\mu B_\nu - D_\nu B_\mu, \quad (8.28b)$$

and the $SU(2)$ gauge field strength tensor is given by

$$\mathbf{F}_{\mu\nu}^n = F_{\mu\nu}^n + gG_\alpha^{-1\lambda} F_\lambda^n F_{\mu\nu}^\alpha, \quad (8.29a)$$

$$F_{\mu\nu}^n = D_\mu F_\nu^n - D_\nu F_\mu^n + g_w \epsilon_{lmn} F_\mu^l F_\nu^m, \quad (8.29b)$$

where g_w is the coupling constant for $SU(2)$ gauge interactions and the coupling constant for $U(1)_Y$ gauge interactions is g'_w .

If there exist Higgs particles in Nature, the Higgs fields is represented by a complex scalar $SU(2)$ doublet,

$$\phi = \begin{pmatrix} \phi^\dagger \\ \phi^0 \end{pmatrix}. \quad (8.30)$$

The hypercharge of Higgs field ϕ is 1.

The Lagrangian \mathcal{L}_0 that describes four kinds of fundamental interactions is given by

$$\begin{aligned}
\mathcal{L}_0 = & -\sum_{j=1}^3 \bar{\psi}_L^{(j)} \gamma^\mu (D_\mu + \frac{i}{2} g'_w B_\mu - i g_w F_\mu) \psi_L^{(j)} \\
& -\sum_{j=1}^3 \bar{e}_R^{(j)} \gamma^\mu (D_\mu + i g'_w B_\mu) e_R^{(j)} \\
& -\sum_{j=1}^3 \bar{q}_L^{(j)a} \gamma^\mu \left((D_\mu - i g_w F_\mu - \frac{i}{6} g'_w B_\mu) \delta_{ab} - i g_c A_\mu^k (\frac{\lambda^k}{2})_{ab} \right) q_L^{(j)b} \\
& -\sum_{j=1}^3 \bar{q}_u^{(j)a} \gamma^\mu \left((D_\mu - i \frac{2}{3} g'_w B_\mu) \delta_{ab} - i g_c A_\mu^k (\frac{\lambda^k}{2})_{ab} \right) q_u^{(j)b} \\
& -\sum_{j=1}^3 \bar{q}_{\theta d}^{(j)a} \gamma^\mu \left((D_\mu + i \frac{1}{3} g'_w B_\mu) \delta_{ab} - i g_c A_\mu^k (\frac{\lambda^k}{2})_{ab} \right) q_{\theta d}^{(j)b} \\
& -\frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} \mathbf{F}_{\mu\nu}^n \mathbf{F}_{\rho\sigma}^n - \frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} \mathbf{B}_{\mu\nu} \mathbf{B}_{\rho\sigma} \\
& -\frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} \mathbf{A}_{\mu\nu}^i \mathbf{A}_{\rho\sigma}^i - \frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} \eta_{2\alpha\beta} F_{\mu\nu}^\alpha F_{\rho\sigma}^\beta \\
& - \left[(D_\mu - \frac{i}{2} g'_w B_\mu - i g_w F_\mu) \phi \right]^\dagger \cdot \left[(D_\mu - \frac{i}{2} g'_w B_\mu - i g_w F_\mu) \phi \right] \\
& -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2 \\
& -\sum_{j=1}^3 f^{(j)} \left(\bar{e}_R^{(j)} \phi^\dagger \psi_L^{(j)} + \bar{\psi}_L^{(j)} \phi e_R^{(j)} \right) \\
& -\sum_{j=1}^3 \left(f_u^{(j)} \bar{q}_L^{(j)a} \bar{\phi} q_u^{(j)a} + f_u^{(j)*} \bar{q}_u^{(j)a} \bar{\phi}^\dagger q_L^{(j)a} \right) \\
& -\sum_{j,k=1}^3 \left(f_d^{(jk)} \bar{q}_L^{(j)a} \phi q_{\theta d}^{(k)a} + f_d^{(jk)*} \bar{q}_{\theta d}^{(k)a} \phi^\dagger q_L^{(j)a} \right),
\end{aligned} \tag{8.31}$$

where

$$\bar{\phi} = i\sigma_2 \phi^* = \begin{pmatrix} \phi^{0\dagger} \\ -\phi \end{pmatrix}. \tag{8.32}$$

The full Lagrangian is given by

$$\mathcal{L} = e^{I(C)} \mathcal{L}_0. \tag{8.33}$$

This Lagrangian describes four kinds of fundamental interactions in Nature. It has $(SU(3) \times SU(2) \times U(1)) \otimes_s$ *Gravitational Gauge Group* symmetry[39]. Four kinds of fundamental interactions are formally unified in this Lagrangian. However, this unification is not a genuine unification. Finally, an important and fundamental problem is that, can we genuine unify four kinds of fundamental interactions in a

single group, in which there is only one coupling constant for all kinds of fundamental interactions? This theory may exist.

9 Classical Limit of Quantum Gravity

It is known that both Newton's theory of gravity and Einstein's general relativity obtain immense achievements in astrophysics and cosmology. Any correct quantum theory of gravity should return to these two theories in classical limit. In this chapter, we will discuss the classical limit of the quantum gauge theory of gravity.

First, we discuss an important problem qualitatively. It is known that, in usual gauge theory, such as QED, the coulomb force between two objects which carry like electric charges is always mutual repulsive. Gravitational gauge theory is also a kind of gauge theory, is the force between two static massive objects attractive or repulsive? For the sake of simplicity, we use Dirac field as an example to discuss this problem. The discussions for other kinds of fields can be proceeded similarly.

Suppose that the gravitational field is very weak, so both the gravitational field and the gravitational coupling constant are first order infinitesimal quantities. Then in leading order approximation, both inertial energy-momentum tensor and gravitational energy-momentum tensor give the same results, which we denoted as

$$T_\alpha^\mu = \bar{\psi}\gamma^\mu\partial_\alpha\psi. \quad (9.1)$$

The time component of the current is

$$T_\alpha^0 = -i\psi^\dagger\partial_\alpha\psi = \psi^\dagger\hat{P}_\alpha\psi. \quad (9.2)$$

Its space integration gives out the energy-momentum of the system. The interaction Lagrangian between Dirac field and gravitational field is given by eq.(6.7). After considering the equation of motion of Dirac field, the coupling between Dirac field and Gravitational gauge field in the leading order is:

$$\mathcal{L}_I \approx gT_\alpha^\mu C_\mu^\alpha. \quad (9.3)$$

The the leading order interaction Hamiltonian density is given by

$$\mathcal{H}_I \approx -\mathcal{L}_I \approx -gT_\alpha^\mu C_\mu^\alpha. \quad (9.4)$$

The equation of motion of gravitational gauge field in the leading order is:

$$\partial_\lambda\partial^\lambda(\eta^{\nu\tau}\eta_{2\alpha\beta}C_\tau^\beta) - \partial^\lambda\partial^\nu(\eta_{2\alpha\beta}C_\lambda^\beta) = -gT_\alpha^\nu. \quad (9.5)$$

As a classical limit approximation, let's consider static gravitational interactions between two static objects. In this case, the leading order component of energy-momentum tensor is T_0^0 , other components of energy-momentum tensor is a first order infinitesimal quantity. So, we only need to consider the equation of motion of $\nu = \alpha = 0$ of eq.(9.5), which now becomes

$$\partial_\lambda \partial^\lambda C_0^0 - \partial^\lambda \partial^\lambda C_\lambda^0 = -gT_0^0. \quad (9.6)$$

For static problems, all time derivatives vanish. Therefore, the above equation is changed into

$$\nabla^2 C_0^0 = -gT_0^0. \quad (9.7)$$

This is just the Newton's equation of gravitational field. Suppose that there is only one point object at the origin of the coordinate system. Because T_0^0 is the negative value of energy density, we can let

$$T_0^0 = -M\delta(\vec{x}). \quad (9.8)$$

Applying

$$\nabla^2 \frac{1}{r} = -4\pi\delta(\vec{x}), \quad (9.9)$$

with $r = |\vec{x}|$, we get

$$C_0^0 = -\frac{gM}{4\pi r}. \quad (9.10)$$

This is just the gravitational potential which is expected in Newton's theory of gravity.

Suppose that there is another point object at the position of point \vec{x} with mass m . The gravitational potential energy between these two objects is that

$$V(r) = \int d^3 \vec{y} \mathcal{H}_I = -g \int d^3 \vec{y} T_{2\ 0}^0(\vec{x}) C_0^0, \quad (9.11)$$

with C_0^0 is the gravitational potential generated by the first point object, and $T_{2\ 0}^0$ is the $(0, 0)$ component of the energy-momentum tensor of the second object,

$$T_{2\ 0}^0(\vec{y}) = -m\delta(\vec{y} - \vec{x}). \quad (9.12)$$

The final result for gravitational potential energy between two point objects is

$$V(r) = -\frac{g^2 Mm}{4\pi r}. \quad (9.13)$$

The gravitational potential energy between two point objects is always negative, which is what expected by Newton's theory of gravity and is the inevitable result of

the attractive nature of gravitational interactions.

The gravitational force that the first point object acts on the second point object is

$$\vec{f} = -\nabla V(r) = -\frac{g^2 Mm}{4\pi r^2} \hat{r}, \quad (9.14)$$

where $\hat{r} = \vec{r} / r$. This is the famous formula of Newton's gravitational force. Therefore, in the classical limit, the gravitational gauge theory can return to Newton's theory of gravity. Besides, from eq.(9.14), we can clearly see that the gravitational interaction force between two point objects is attractive.

Now, we want to ask a problem: why in QED, the force between two like electric charges is always repulsive, while in gravitational gauge theory, the force between two like gravitational charges is always attractive? A simple answer to this fundamental problem is that the attractive nature of the gravitational force is an inevitable result of the global Lorentz symmetry of the system. Because of the requirement of global Lorentz symmetry, the Lagrangian function given by eq.(4.20) must use $\eta_{2\alpha\beta}$, can not use the ordinary $\delta_{\alpha\beta}$. It can be easily prove that, if we use $\delta_{\alpha\beta}$ instead of $\eta_{2\alpha\beta}$ in eq.(4.20), the Lagrangian of pure gravitational gauge field is not invariant under global Lorentz transformation. On the other hand, if we use $\delta_{\alpha\beta}$ instead of $\eta_{2\alpha\beta}$ in eq.(4.20), the gravitational force will be repulsive which obviously contradicts with experiment results. In QED, δ_{ab} is used to construct the Lagrangian for electromagnetic fields, therefore, the interaction force between two like electric charges is always repulsive.

One fundamental influence of using the metric $\eta_{2\alpha\beta}$ in the Lagrangian of pure gravitational field is that the kinematic energy term of gravitation field C_μ^0 is always negative. This result is novels, but it is not surprising, for gravitational interaction energy is always negative. In a meaning, it is the reflection of the negative nature of graviton's kinematic energy. Though the kinematic energy term of gravitation field C_μ^0 is always negative, the kinematic energy term of gravitation field C_μ^i is always positive. The negative energy problem of graviton does not cause any trouble in quantum gauge theory of gravity. Contrarily, it will help us to understand some puzzle phenomena of Nature. From theoretical point of view, the negative nature of graviton's kinematic energy is essentially an inevitable result of global Lorentz symmetry. Global Lorentz symmetry of the system, attractive nature of gravitational interaction force and negative nature of graviton's kinematical energy are essentially related to each other, and they have the same origin in nature. We will return to discuss the negative energy problem again later.

In general relativity, gravitational field obeys Einstein field equation, which is

usually written in the following form,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \lambda g_{\mu\nu} = -8\pi GT_{\mu\nu}, \quad (9.15)$$

where $R_{\mu\nu}$ is Ricci tensor, R is curvature, G is Newton gravitational constant and λ is cosmology constant. The classical limit of Einstein field equation is

$$\nabla^2 g_{00} = -8\pi GT_{00}. \quad (9.16)$$

Compare this equation with eq.(9.7) and use eq.(5.24), we get

$$g^2 = 4\pi G. \quad (9.17)$$

In order to get eq.(9.17), the following relations are used

$$T_{00} = -T_0^0, \quad g_{00} \simeq -(1 + 2gC_0^0). \quad (9.18)$$

In general relativity, Einstein field equation transforms covariantly under general coordinates transformation, in other words, it is a general covariant equation. In gravitational gauge theory, the system has local gravitational gauge symmetry. From mathematical point of view, general coordinates transformation is equivalent to local gravitational gauge transformation. Therefore, it seems that two theories have the same symmetry. On the other hand, both theories have global Lorentz symmetry.

In the previous discussions, we have given out the relations between space-time metric and gravitational gauge fields, which is shown in eq.(5.24). For scalar field, if the gravitational field is not so strong and its variations with space-time is not so great, we can select a local inertial reference system where gravity is completely shielded, which is shown in eq.(5.23), where we can not directly see any coupling between gravitational gauge field and scalar field. For a general macroscopic object, its dynamical properties are more like real scalar field. Therefore, in macroscopic world, we can always select a local inertial reference system where all macroscopic effects of gravitational interactions are put into the structure space-time, which is the main point of view of general relativity. Now, let's take this point of view to study structures of space-time in the classical and macroscopic limit.

Let's omit Lagrangian of pure gravitational gauge fields, then eq.(5.23) is changed into

$$\mathcal{L} = -\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \frac{m^2}{2}\phi^2. \quad (9.19)$$

This is the Lagrangian for curved space-time. The local inertial reference system is given by following coordinates transformation,

$$dx^\mu \rightarrow dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu, \quad (9.20)$$

where $\frac{\partial x'^\mu}{\partial x^\nu}$ is given by,

$$\frac{\partial x'^\mu}{\partial x^\nu} = (G^{-1})^\mu_\nu. \quad (9.21)$$

Define matrix G as

$$G = (G_\mu^\alpha) = (\delta_\mu^\alpha - gC_\mu^\alpha). \quad (9.22)$$

A simple form for matrix G is

$$G = I - gC, \quad (9.23)$$

where I is a unit matrix and $C = (C_\mu^\alpha)$. Therefore,

$$G^{-1} = \frac{1}{I - gC}. \quad (9.24)$$

If gravitational gauge fields are weak enough, we have

$$G^{-1} = \sum_{n=0}^{\infty} (gC)^n. \quad (9.25)$$

G^{-1} is the inverse matrix of G , it satisfies

$$(G^{-1})^\mu_\beta G_\mu^\alpha = \delta_\beta^\alpha, \quad (9.26)$$

$$G_\mu^\alpha (G^{-1})^\nu_\alpha = \delta_\mu^\nu. \quad (9.27)$$

Using all these relations, we can prove that

$$\begin{aligned} g^{\alpha\beta} \rightarrow g'^{\alpha\beta} &= \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} g^{\mu\nu} \\ &= \eta^{\alpha\beta}. \end{aligned} \quad (9.28)$$

Therefore, under this coordinates transformation, the space-time metric becomes flat, in other words, we go into local inertial reference system. In this local inertial reference system, the Lagrangian eq.(9.19) becomes

$$\mathcal{L} = -\frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \frac{m^2}{2}\phi^2. \quad (9.29)$$

Eq.(9.29) is just the Lagrangian for real scalar fields in flat Minkowski space-time.

Define covariant metric tensor $g_{\alpha\beta}$ as

$$g_{\alpha\beta} \triangleq \eta_{\mu\nu} (G^{-1})_{\alpha}^{\mu} (G^{-1})_{\beta}^{\nu}. \quad (9.30)$$

It can be easily proved that

$$g_{\alpha\beta} g^{\beta\gamma} = \delta_{\alpha}^{\gamma}, \quad (9.31)$$

$$g^{\alpha\beta} g_{\beta\gamma} = \delta_{\gamma}^{\alpha}. \quad (9.32)$$

The affine connection $\Gamma_{\mu\nu}^{\lambda}$ is defined by

$$\Gamma_{\mu\nu}^{\lambda} = \frac{1}{2} g^{\lambda\sigma} \left(\frac{\partial g_{\mu\sigma}}{\partial x^{\nu}} + \frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} \right). \quad (9.33)$$

Using the following relation,

$$g F_{\rho\sigma}^{\lambda} = G_{\rho}^{\nu} G_{\sigma}^{\mu} [(G^{-1} \partial_{\mu} G)_{\nu}^{\lambda} - (G^{-1} \partial_{\nu} G)_{\mu}^{\lambda}], \quad (9.34)$$

where $F_{\rho\sigma}^{\lambda}$ is the component field strength of gravitational gauge field, we get

$$\begin{aligned} \Gamma_{\mu\nu}^{\lambda} = & -\frac{1}{2} [(G^{-1} \partial_{\mu} G)_{\nu}^{\lambda} + (G^{-1} \partial_{\nu} G)_{\mu}^{\lambda}] \\ & + \frac{1}{2} g \eta^{\alpha_1 \beta_1} \eta_{\alpha\beta} F_{\mu_1 \beta_1}^{\rho} G_{\alpha_1}^{\lambda} G_{\rho}^{-1\alpha} (G_{\nu}^{-1\beta} G_{\mu}^{-1\mu_1} + G_{\mu}^{-1\beta} G_{\nu}^{-1\mu_1}). \end{aligned} \quad (9.35)$$

From this expression, we can see that, if there is no gravity in space-time, that is

$$C_{\mu}^{\alpha} = 0 \quad , \quad F_{\mu\nu}^{\lambda} = 0, \quad (9.36)$$

then the affine connection $\Gamma_{\mu\nu}^{\lambda}$ will vanish, which is what we expect in general relativity.

The curvature tensor $R_{\mu\nu\kappa}^{\lambda}$ is defined by

$$R_{\mu\nu\kappa}^{\lambda} \triangleq \partial_{\kappa} \Gamma_{\mu\nu}^{\lambda} - \partial_{\nu} \Gamma_{\mu\kappa}^{\lambda} + \Gamma_{\mu\nu}^{\eta} \Gamma_{\kappa\eta}^{\lambda} - \Gamma_{\mu\kappa}^{\eta} \Gamma_{\nu\eta}^{\lambda}, \quad (9.37)$$

the Ricci tensor $R_{\mu\kappa}$ is defined by

$$R_{\mu\kappa} \triangleq R_{\mu\lambda\kappa}^{\lambda}, \quad (9.38)$$

and the curvature scalar R is defined by

$$R \triangleq g^{\mu\kappa} R_{\mu\kappa}. \quad (9.39)$$

The explicit expression for Ricci tensor $R_{\mu\kappa}$ is

$$\begin{aligned}
R_{\mu\kappa} = & -(\partial_\kappa \partial_\mu G \cdot G^{-1})_\alpha^\alpha + 2(\partial_\kappa G \cdot G^{-1} \cdot \partial_\mu G \cdot G^{-1})_\alpha^\alpha \\
& + \eta^{\rho\sigma} \eta_{\alpha\beta} (\partial_\mu G \cdot G^{-1})_\rho^\alpha (\partial_\kappa G \cdot G^{-1})_\sigma^\beta \\
& + \frac{1}{2} g^{\lambda\nu} \eta_{\alpha\beta} (G^{-1} \cdot \partial_\nu G \cdot G^{-1} \cdot \partial_\lambda G \cdot G^{-1})_\mu^\alpha G_\kappa^{-1\beta} \\
& - \frac{1}{2} g^{\lambda\nu} \eta_{\alpha\beta} (G^{-1} \cdot \partial_\nu \partial_\lambda G \cdot G^{-1})_\mu^\alpha G_\kappa^{-1\beta} \\
& + \frac{1}{2} g^{\lambda\nu} \eta_{\alpha\beta} (G^{-1} \cdot \partial_\lambda G \cdot G^{-1} \cdot \partial_\nu G \cdot G^{-1})_\mu^\alpha G_\kappa^{-1\beta} \\
& + \frac{1}{2} g^{\lambda\nu} \eta_{\alpha\beta} G_\mu^{-1\alpha} (G^{-1} \cdot \partial_\nu G \cdot G^{-1} \cdot \partial_\lambda G \cdot G^{-1})_\kappa^\beta \\
& - \frac{1}{2} g^{\lambda\nu} \eta_{\alpha\beta} G_\mu^{-1\alpha} (G^{-1} \cdot \partial_\nu \partial_\lambda G \cdot G^{-1})_\kappa^\beta \\
& + \frac{1}{2} g^{\lambda\nu} \eta_{\alpha\beta} G_\mu^{-1\alpha} (G^{-1} \cdot \partial_\lambda G \cdot G^{-1} \cdot \partial_\nu G \cdot G^{-1})_\kappa^\beta \\
& + g^{\lambda\nu} \eta_{\alpha\beta} (G^{-1} \cdot \partial_\nu G \cdot G^{-1})_\mu^\alpha (G^{-1} \cdot \partial_\lambda G \cdot G^{-1})_\kappa^\beta \\
& - \frac{1}{2} (G^{-1} \cdot \partial_\kappa G \cdot G^{-1} \cdot \partial_\lambda G)_\mu^\lambda - \frac{1}{2} (G^{-1} \cdot \partial_\lambda G \cdot G^{-1} \cdot \partial_\kappa G)_\mu^\lambda + \frac{1}{2} (G^{-1} \cdot \partial_\kappa \partial_\lambda G)_\mu^\lambda \\
& - \eta^{\rho\sigma} \eta_{\alpha\beta} G_\rho^\lambda (G^{-1} \cdot \partial_\lambda G \cdot G^{-1})_\mu^\alpha (\partial_\kappa G \cdot G^{-1})_\sigma^\beta \\
& - \frac{1}{2} \eta^{\rho\sigma} \eta_{\alpha\beta} G_\rho^\lambda G_\mu^{-1\alpha} (\partial_\kappa G \cdot G^{-1} \cdot \partial_\lambda G \cdot G^{-1})_\sigma^\beta \\
& + \frac{1}{2} \eta^{\rho\sigma} \eta_{\alpha\beta} G_\rho^\lambda G_\mu^{-1\alpha} (\partial_\kappa \partial_\lambda G \cdot G^{-1})_\sigma^\beta \\
& - \frac{1}{2} \eta^{\rho\sigma} \eta_{\alpha\beta} G_\rho^\lambda G_\mu^{-1\alpha} (\partial_\lambda G \cdot G^{-1} \cdot \partial_\kappa G \cdot G^{-1})_\sigma^\beta \\
& - \frac{1}{2} (G^{-1} \cdot \partial_\nu G \cdot G^{-1} \cdot \partial_\mu G)_\kappa^\nu - \frac{1}{2} (G^{-1} \cdot \partial_\mu G \cdot G^{-1} \cdot \partial_\nu G)_\kappa^\nu + \frac{1}{2} (G^{-1} \cdot \partial_\nu \partial_\mu G)_\kappa^\nu \\
& - \eta^{\rho\sigma} \eta_{\alpha\beta} G_\sigma^\nu (G^{-1} \cdot \partial_\mu G \cdot G^{-1})_\kappa^\alpha (\partial_\nu G \cdot G^{-1})_\rho^\beta \\
& - \frac{1}{2} \eta^{\rho\sigma} \eta_{\alpha\beta} G_\sigma^\nu G_\kappa^{-1\alpha} (\partial_\nu G \cdot G^{-1} \cdot \partial_\mu G \cdot G^{-1})_\rho^\beta \\
& + \frac{1}{2} \eta^{\rho\sigma} \eta_{\alpha\beta} G_\sigma^\nu G_\kappa^{-1\alpha} (\partial_\nu \partial_\mu G \cdot G^{-1})_\rho^\beta - \frac{1}{2} \eta^{\rho\sigma} \eta_{\alpha\beta} G_\sigma^\nu G_\kappa^{-1\alpha} (\partial_\mu G \cdot G^{-1} \cdot \partial_\nu G \cdot G^{-1})_\rho^\beta \\
& + \frac{1}{2} \eta_{\alpha\beta} \eta^{\alpha_1\beta_1} G_{\beta_1}^\nu (\partial_\nu G \cdot G^{-1})_{\alpha_1}^\alpha (G^{-1} \cdot \partial_\mu G \cdot G^{-1})_{\beta_1}^\beta \\
& + \frac{1}{2} \eta_{\alpha\beta} \eta^{\alpha_1\beta_1} G_{\beta_1}^\nu (\partial_\nu G \cdot G^{-1})_{\alpha_1}^\alpha (G^{-1} \cdot \partial_\kappa G \cdot G^{-1})_{\beta_1}^\beta \\
& - \frac{1}{4} \eta_{\alpha\beta} \eta^{\alpha_1\beta_1} (\partial_\kappa G \cdot G^{-1})_{\alpha_1}^\alpha (\partial_\mu G \cdot G^{-1})_{\beta_1}^\beta \\
& - \frac{1}{4} \eta_{\alpha\beta} \eta^{\alpha_1\beta_1} G_{\beta_1}^\nu (\partial_\kappa G \cdot G^{-1})_{\alpha_1}^\alpha (G^{-1} \cdot \partial_\nu G \cdot G^{-1})_{\beta_1}^\beta \\
& - \frac{1}{4} \eta_{\alpha\beta} \eta^{\alpha_1\beta_1} G_{\alpha_1}^\lambda (G^{-1} \cdot \partial_\lambda G \cdot G^{-1})_{\beta_1}^\alpha (\partial_\mu G \cdot G^{-1})_{\beta_1}^\beta \\
& - \frac{1}{4} \eta_{\alpha\beta} \eta^{\alpha_1\beta_1} G_{\alpha_1}^\lambda G_{\beta_1}^\nu (G^{-1} \cdot \partial_\lambda G \cdot G^{-1})_{\beta_1}^\alpha (G^{-1} \cdot \partial_\nu G \cdot G^{-1})_{\beta_1}^\beta \\
& - \frac{g}{2} \eta_{\alpha\beta} \eta^{\alpha_3\beta_3} F_{\mu_1\beta_1}^\rho G_{\beta_3}^\nu G_\rho^{-1\alpha} G_\kappa^{-1\beta} G_\mu^{-1\mu_1} (\partial_\nu G \cdot G^{-1})_{\alpha_3}^{\beta_1} \\
& - \frac{g}{2} \eta_{\alpha\beta} \eta^{\alpha_3\beta_3} F_{\mu_1\beta_1}^\rho G_{\beta_3}^\nu G_\rho^{-1\alpha} G_\kappa^{-1\beta} G_\mu^{-1\mu_1} (\partial_\nu G \cdot G^{-1})_{\alpha_3}^{\beta_1} \\
& - \frac{g}{2} F_{\alpha\beta}^\rho G_\rho^{-1\alpha} (G^{-1} \cdot \partial_\mu G \cdot G^{-1})_\kappa^\beta - \frac{g}{2} F_{\alpha\beta}^\rho G_\rho^{-1\alpha} (G^{-1} \cdot \partial_\kappa G \cdot G^{-1})_\mu^\beta \\
& + \frac{g}{4} \eta^{\alpha_3\mu_1} \eta_{\alpha\beta} F_{\mu_1\beta_1}^\rho G_\rho^{-1\alpha} G_\mu^{-1\beta} (\partial_\kappa G \cdot G^{-1})_{\alpha_3}^{\beta_1} + \frac{g}{4} F_{\alpha\beta}^\rho (G^{-1} \cdot \partial_\kappa G \cdot G^{-1})_\rho^\beta G_\mu^{-1\alpha} \\
& + \frac{g}{4} \eta^{\alpha_3\mu_1} \eta_{\alpha\beta} F_{\mu_1\beta_1}^\rho G_\rho^{-1\alpha} G_\mu^{-1\beta} G_{\alpha_3}^\lambda (G^{-1} \cdot \partial_\lambda G \cdot G^{-1})_{\beta_1}^{\beta_1} \\
& + \frac{g}{4} F_{\alpha\beta}^\rho (G^{-1} \cdot \partial_\rho G \cdot G^{-1})_\kappa^\beta G_\mu^{-1\alpha} + \frac{g}{4} F_{\alpha\beta}^\rho (G^{-1} \cdot \partial_\rho G \cdot G^{-1})_\mu^\beta G_\kappa^{-1\alpha} \\
& + \frac{g}{4} \eta^{\alpha_3\mu_1} \eta_{\alpha\beta} F_{\mu_1\beta_1}^\rho G_\rho^{-1\alpha} G_\kappa^{-1\beta} (\partial_\mu G \cdot G^{-1})_{\alpha_3}^{\beta_1} + \frac{g}{4} F_{\alpha\beta}^\rho (G^{-1} \cdot \partial_\mu G \cdot G^{-1})_\rho^\beta G_\kappa^{-1\alpha} \\
& + \frac{g}{4} \eta^{\alpha_3\mu_1} \eta_{\alpha\beta} F_{\mu_1\beta_1}^\rho G_\rho^{-1\alpha} G_\kappa^{-1\beta} G_{\alpha_3}^\lambda (G^{-1} \cdot \partial_\lambda G \cdot G^{-1})_{\beta_1}^{\beta_1} \\
& + \frac{g^2}{2} \eta^{\sigma\beta_1} \eta_{\alpha_2\beta_2} F_{\alpha\sigma}^\rho F_{\mu_2\beta_1}^{\rho_2} G_\rho^{-1\alpha} G_\mu^{-1\alpha_2} (G_\mu^{-1\beta_2} G_\kappa^{-1\mu_2} + G_\kappa^{-1\beta_2} G_\mu^{-1\mu_2}) \\
& - \frac{g^2}{4} \eta_{\alpha\beta} \eta_{\alpha_2\beta_2} \eta^{\alpha_1\beta_1} \eta^{\mu_1\sigma_2} F_{\mu_1\beta_1}^\rho F_{\sigma_2\alpha_1}^{\rho_1} G_\rho^{-1\alpha} G_\mu^{-1\alpha_2} G_\kappa^{-1\beta} G_\mu^{-1\beta_2} \\
& - \frac{g^2}{4} \eta_{\alpha\beta} \eta^{\alpha_1\beta_1} F_{\mu_1\beta_1}^\rho F_{\mu_3\alpha_1}^{\rho_1} G_\rho^{-1\alpha} G_{\rho_1}^{-1\mu_1} G_\kappa^{-1\beta} G_\mu^{-1\mu_3} \\
& - \frac{g^2}{4} \eta_{\alpha_2\beta_2} \eta^{\alpha_1\beta_1} F_{\mu_1\beta_1}^\rho F_{\alpha\alpha_1}^{\rho_1} G_\rho^{-1\alpha} G_{\rho_1}^{-1\alpha_2} G_\kappa^{-1\mu_1} G_\mu^{-1\beta_2} \\
& - \frac{g^2}{4} \eta_{\alpha\beta} \eta^{\alpha_1\beta_1} F_{\mu_1\beta_1}^\rho F_{\mu_3\alpha_1}^{\rho_1} G_\rho^{-1\alpha} G_{\rho_1}^{-1\beta} G_\kappa^{-1\mu_1} G_\mu^{-1\mu_3}
\end{aligned} \tag{9.40}$$

The explicit expression for scalar curvature R is

$$\begin{aligned}
R = & 4g^{\mu\kappa}(\partial_\mu G \cdot G^{-1} \cdot \partial_\kappa G \cdot G^{-1})_\alpha^\alpha - 2g^{\mu\kappa}(\partial_\mu \partial_\kappa G \cdot G^{-1})_\alpha^\alpha \\
& + \frac{3}{2}\eta^{\rho\sigma}\eta_{\alpha\beta}g^{\mu\kappa}(\partial_\mu G \cdot G^{-1})_\rho^\alpha(\partial_\kappa G \cdot G^{-1})_\sigma^\beta - 2\eta^{\alpha\beta}G_\beta^\kappa(\partial_\kappa G \cdot G^{-1} \cdot \partial_\lambda G)_\alpha^\lambda \\
& - 2\eta^{\alpha\beta}G_\beta^\kappa(\partial_\lambda G \cdot G^{-1} \cdot \partial_\kappa G)_\alpha^\lambda + 2\eta^{\alpha\beta}G_\beta^\kappa(\partial_\kappa \partial_\lambda G)_\alpha^\lambda \\
& - \frac{5}{2}\eta_{\alpha\beta}\eta^{\rho\sigma}\eta^{\mu\nu}G_\nu^\kappa G_\rho^\lambda(\partial_\lambda G \cdot G^{-1})_\mu^\alpha(\partial_\kappa G \cdot G^{-1})_\sigma^\beta \\
& + \eta_{\alpha\beta}\eta^{\alpha_1\beta_1}\eta^{\mu_1\nu_1}G_{\beta_1}^\nu G_{\mu_1}^\mu(\partial_\nu G \cdot G^{-1})_{\alpha_1}^\alpha(\partial_\mu G \cdot G^{-1})_{\nu_1}^\beta \\
& - 2g\eta^{\alpha\beta}F_{\alpha_1\beta_1}^\rho G_{\beta_1}^\nu G_{\rho_1}^{-1\alpha_1}(\partial_\nu G \cdot G^{-1})_{\alpha_1}^{\beta_1} \\
& + g\eta^{\alpha\mu}F_{\mu\beta}^\kappa(\partial_\kappa G \cdot G^{-1})_\alpha^\beta \\
& + g\eta^{\alpha\mu}F_{\alpha\beta}^\rho G_\mu^\kappa(G^{-1} \cdot \partial_\kappa G \cdot G^{-1})_\rho^\beta \\
& + g^2\eta^{\beta\beta_1}F_{\alpha\beta}^\rho F_{\alpha_1\beta_1}^{\rho_1}G_\rho^{-1\alpha}G_{\rho_1}^{-1\alpha_1} \\
& - \frac{g^2}{2}\eta_{\alpha\beta}\eta^{\alpha_1\beta_1}\eta^{\mu\sigma}F_{\mu\beta_1}^\rho F_{\sigma\alpha_1}^{\rho_1}G_\rho^{-1\alpha}G_{\rho_1}^{-1\beta} \\
& - \frac{g^2}{2}\eta^{\alpha\beta}F_{\mu\beta}^\rho F_{\alpha_1\alpha}^{\rho_1}G_\rho^{-1\alpha_1}G_{\rho_1}^{-1\mu}
\end{aligned} \tag{9.41}$$

From these expressions, we can see that, if there is no gravity, that is C_μ^α vanishes, then $R_{\mu\nu}$ and R all vanish. It means that, if there is gravity, the space-time is flat, which is what we expected in general relativity.

Now, let's discuss some transformation properties of these tensors under general coordinates transformation. Make a special kind of local coordinates translation,

$$x^\mu \rightarrow x'^\mu = x^\mu + \epsilon^\mu(x'). \tag{9.42}$$

Under this transformation, the covariant derivative and gravitational gauge fields transform as

$$D_\mu(x) \rightarrow D'_\mu(x') = \hat{U}_\epsilon(x')D_\mu(x')\hat{U}_\epsilon^{-1}(x'), \tag{9.43}$$

$$C_\mu(x) \rightarrow C'_\mu(x') = \hat{U}_\epsilon(x')C_\mu(x')\hat{U}_\epsilon^{-1}(x') + \frac{i}{g}\hat{U}_\epsilon(x')\left(\frac{\partial}{\partial x'^\mu}\hat{U}_\epsilon^{-1}(x')\right). \tag{9.44}$$

It can be proved that

$$G_\mu^\alpha(x) \rightarrow G'^\alpha_\mu(x') = \Lambda^\alpha_\beta G_\mu^\beta(x), \tag{9.45}$$

where

$$\Lambda^\alpha_\beta = \frac{\partial x'^\alpha}{\partial x^\beta}. \tag{9.46}$$

We can see that Lorentz index μ does not take part in transformation. In fact, all Lorentz indexes do not take part in this kind of transformation. Therefore,

$$\eta^{\mu\nu} \rightarrow \eta'^{\mu\nu} = \eta^{\mu\nu}. \tag{9.47}$$

Using all these relations, we can prove that

$$g^{\alpha\beta}(x) \rightarrow g'^{\alpha\beta}(x') = \Lambda^\alpha_{\alpha_1}\Lambda^\beta_{\beta_1}g^{\alpha_1\beta_1}(x), \tag{9.48}$$

$$R_{\alpha\beta\gamma\delta}(x) \rightarrow R_{\alpha\beta\gamma\delta}(x') = \Lambda_{\alpha}^{\alpha_1} \Lambda_{\beta}^{\beta_1} \Lambda_{\gamma}^{\gamma_1} \Lambda_{\delta}^{\delta_1} g_{\alpha_1\beta_1\gamma_1\delta_1}(x), \quad (9.49)$$

$$R_{\alpha\beta}(x) \rightarrow R'_{\alpha\beta}(x') = \Lambda_{\alpha}^{\alpha_1} \Lambda_{\beta}^{\beta_1} g_{\alpha_1\beta_1}(x), \quad (9.50)$$

$$R(x) \rightarrow R'(x') = R(x). \quad (9.51)$$

These transformation properties are just what we expected in general relativity.

In vacuum, there is no matter field and the energy momentum tensor of matter fields vanishes. Suppose that the self-energy of gravitational field is small enough to be neglected, that is

$$T_{\mu\nu} \approx 0, \quad (9.52)$$

then Einstein field equation becomes

$$G_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0. \quad (9.53)$$

We will prove that, in the classical limit of gravitational gauge field theory, the Einstein field equation of vacuum holds in the first order approximation of gC_{μ}^{α} . Using relation eq.(9.22), we get

$$\partial_{\mu}G = -g\partial_{\mu}C, \quad (9.54)$$

which is a first order quantity of gC_{μ}^{α} . In first order approximations, eq.(9.40) and eq.(9.41) are changed into

$$\begin{aligned} R_{\mu\kappa} = & g\partial_{\kappa}\partial_{\mu}C_{\alpha}^{\alpha} - \frac{g}{2}\partial_{\kappa}\partial_{\lambda}C_{\mu}^{\lambda} - \frac{g}{2}\partial_{\mu}\partial_{\lambda}C_{\kappa}^{\lambda} \\ & + \frac{g}{2}\eta^{\lambda\nu}\eta_{\alpha\kappa}\partial_{\nu}\partial_{\lambda}C_{\mu}^{\alpha} + \frac{g}{2}\eta^{\lambda\nu}\eta_{\alpha\mu}\partial_{\nu}\partial_{\lambda}C_{\kappa}^{\alpha} \\ & - \frac{g}{2}\eta^{\alpha\sigma}\eta_{\mu\beta}\partial_{\kappa}\partial_{\alpha}C_{\sigma}^{\beta} - \frac{g}{2}\eta^{\alpha\sigma}\eta_{\kappa\beta}\partial_{\mu}\partial_{\alpha}C_{\sigma}^{\beta} + o((gC)^2), \end{aligned} \quad (9.55)$$

$$R = 2g\eta^{\alpha\beta}\partial_{\alpha}\partial_{\beta}C_{\sigma}^{\sigma} - 2g\eta^{\alpha\beta}\partial_{\beta}\partial_{\sigma}C_{\alpha}^{\sigma} + o((gC)^2). \quad (9.56)$$

The equation of motion eq.(9.5) of gravitational gauge fields gives out the following constraint on gravitational gauge field in vacuum:

$$\partial^{\lambda}\partial_{\lambda}C_{\tau}^{\beta} = \partial^{\lambda}\partial_{\tau}C_{\lambda}^{\beta}. \quad (9.57)$$

Set $\beta = \tau$, eq.(9.57) gives out

$$\partial^{\lambda}\partial_{\lambda}C_{\beta}^{\beta} = \partial^{\lambda}\partial_{\beta}C_{\lambda}^{\beta}. \quad (9.58)$$

Using these two constraints, we can prove that $R_{\mu\nu}$ and R given by eqs.(9.55-56) vanish:

$$R_{\mu\nu} = 0, \quad (9.59)$$

$$R = 0. \quad (9.60)$$

Therefore, Einstein field equation of vacuum indeed holds in first order approximation of gC_μ^α , which is indeed a second order infinitesimal quantity.

Equivalence principle is one of the most important fundamental principles of general relativity. But, as we have studied in previous chapters, the inertial energy-momentum is not equivalent to the gravitational energy-momentum in gravitational gauge theory. This result is an inevitable result of gauge principle. But all these differences are caused by gravitational gauge field. In leading term approximation, the inertial energy-momentum tensor $T_{i\alpha}^\mu$ is the same as the gravitational energy-momentum tensor $T_{g\alpha}^\mu$. Because gravitational coupling constant g is extremely small and the strength of gravitational field is also weak, it is hard to detect the difference between inertial mass and gravitational mass. Using gravitational gauge field theory, we can calculate the difference of inertial mass and gravitational mass for different kinds of matter and help us to test the validity of equivalence principle. This is a fundamental problem which will help us to understand the nature of gravitational interactions. We will return to this problem in chapter 12.

Through above discussions, we can make following two important conclusions: 1) the leading order approximation of gravitational gauge theory gives out Newton's theory of gravity; 2) in the first order approximation, equation of motion of gravitational gauge fields in vacuum gives out Einstein field equation of vacuum.

10 Path Integral Quantization of Gravitational Gauge Fields

For the sake of simplicity, in this chapter and the next chapter, we only discuss pure gravitational gauge field. For pure gravitational gauge field, its Lagrangian function is

$$\mathcal{L} = -\frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}\eta_{2\alpha\beta}e^{I(C)}F_{\mu\nu}^\alpha F_{\rho\sigma}^\beta. \quad (10.1)$$

Its space-time integration gives out the action of the system

$$S = \int d^4x \mathcal{L}. \quad (10.2)$$

This action has local gravitational gauge symmetry. Gravitational gauge field C_μ^α has $4 \times 4 = 16$ degrees of freedom. But, if gravitons are massless, the system has only $2 \times 4 = 8$ degrees of freedom. There are gauge degrees of freedom in the theory. Because only physical degrees of freedom can be quantized, in order to quantize the system, we have to introduce gauge conditions to eliminate un-physical degrees of

freedom. For the sake of convenience, we take temporal gauge conditions

$$C_0^\alpha = 0, \quad (\alpha = 0, 1, 2, 3). \quad (10.3)$$

In temporal gauge, the generating functional $W[J]$ is given by

$$W[J] = N \int [\mathcal{D}C] \left(\prod_{\alpha,x} \delta(C_0^\alpha(x)) \right) \exp \left\{ i \int d^4x (\mathcal{L} + J_\alpha^\mu C_\mu^\alpha) \right\} \quad (10.4)$$

where N is the normalization constant, J_α^μ is a fixed external source and $[\mathcal{D}C]$ is the integration measure,

$$[\mathcal{D}C] = \prod_{\mu=0}^3 \prod_{\alpha=0}^3 \prod_j \left(\varepsilon dC_\mu^\alpha(\tau_j) / \sqrt{2\pi i \hbar} \right). \quad (10.5)$$

We use this generation functional as our starting point of the path integral quantization of gravitational gauge field.

Generally speaking, the action of the system has local gravitational gauge symmetry, but the gauge condition has no local gravitational gauge symmetry. If we make a local gravitational gauge transformations, the action of the system is kept unchanged while gauge condition will be changed. Therefore, through local gravitational gauge transformation, we can change one gauge condition to another gauge condition. The most general gauge condition is

$$f^\alpha(C(x)) - \varphi^\alpha(x) = 0, \quad (10.6)$$

where $\varphi^\alpha(x)$ is an arbitrary space-time function. The Fadeev-Popov determinant $\Delta_f(C)$ [28] is defined by

$$\Delta_f^{-1}(C) \equiv \int [\mathcal{D}g] \prod_{x,\alpha} \delta(f^\alpha({}^g C(x)) - \varphi^\alpha(x)), \quad (10.7)$$

where g is an element of gravitational gauge group, ${}^g C$ is the gravitational gauge field after gauge transformation g and $[\mathcal{D}g]$ is the integration measure on gravitational gauge group

$$[\mathcal{D}g] = \prod_x d^4 \epsilon(x), \quad (10.8)$$

where $\epsilon(x)$ is the transformation parameter of \hat{U}_ϵ . Both $[\mathcal{D}g]$ and $[\mathcal{D}C]$ are not invariant under gravitational gauge transformation. Suppose that,

$$[\mathcal{D}(gg')] = J_1(g') [\mathcal{D}g], \quad (10.9)$$

$$[\mathcal{D} {}^g C] = J_2(g)[\mathcal{D} C]. \quad (10.10)$$

$J_1(g)$ and $J_2(g)$ satisfy the following relations

$$J_1(g) \cdot J_1(g^{-1}) = 1, \quad (10.11)$$

$$J_2(g) \cdot J_2(g^{-1}) = 1. \quad (10.12)$$

It can be proved that, under gravitational gauge transformations, the Fadeev-Popov determinant transforms as

$$\Delta_f^{-1}(g' C) = J_1^{-1}(g') \Delta_f^{-1}(C). \quad (10.13)$$

Insert eq.(10.7) into eq.(10.4), we get

$$\begin{aligned} W[J] = & N \int [\mathcal{D} g] \int [\mathcal{D} C] \left[\prod_{\alpha, y} \delta(C_0^\alpha(y)) \right] \cdot \Delta_f(C) \\ & \cdot \left[\prod_{\beta, z} \delta(f^\beta({}^g C(z)) - \varphi^\beta(z)) \right] \cdot \exp \left\{ i \int d^4 x (\mathcal{L} + J_\alpha^\mu C_\mu^\alpha) \right\}. \end{aligned} \quad (10.14)$$

Make a gravitational gauge transformation,

$$C(x) \rightarrow g^{-1} C(x), \quad (10.15)$$

then,

$${}^g C(x) \rightarrow g g^{-1} C(x). \quad (10.16)$$

After this transformation, the generating functional is changed into

$$\begin{aligned} W[J] = & N \int [\mathcal{D} g] \int [\mathcal{D} C] J_1(g) J_2(g^{-1}) \cdot \left[\prod_{\alpha, y} \delta(g^{-1} C_0^\alpha(y)) \right] \cdot \Delta_f(C) \\ & \cdot \left[\prod_{\beta, z} \delta(f^\beta(C(z)) - \varphi^\beta(z)) \right] \cdot \exp \left\{ i \int d^4 x (\mathcal{L} + J_\alpha^\mu \cdot g^{-1} C_\mu^\alpha) \right\}. \end{aligned} \quad (10.17)$$

Suppose that the gauge transformation $g_0(C)$ transforms general gauge condition $f^\beta(C) - \varphi^\beta = 0$ to temporal gauge condition $C_0^\alpha = 0$, and suppose that this transformation $g_0(C)$ is unique. Then two δ -functions in eq.(10.17) require that the integration on gravitational gauge group must be in the neighborhood of $g_0^{-1}(C)$. Therefore eq.(10.17) is changed into

$$\begin{aligned} W[J] = & N \int [\mathcal{D} C] \Delta_f(C) \cdot \left[\prod_{\beta, z} \delta(f^\beta(C(z)) - \varphi^\beta(z)) \right] \\ & \cdot \exp \left\{ i \int d^4 x (\mathcal{L} + J_\alpha^\mu \cdot g_0 C_\mu^\alpha) \right\} \\ & \cdot J_1(g_0^{-1}) J_2(g_0) \cdot \int [\mathcal{D} g] \left[\prod_{\alpha, y} \delta(g^{-1} C_0^\alpha(y)) \right]. \end{aligned} \quad (10.18)$$

The last line in eq.(10.18) will cause no trouble in renormalization, and if we consider the contribution from ghost fields which will be introduced below, it will become a quantity which is independent of gravitational gauge field. So, we put it into normalization constant N and still denote the new normalization constant as N . We also change $J_\alpha^\mu g^0 C_\mu^\alpha$ into $J_\alpha^\mu C_\mu^\alpha$, this will cause no trouble in renormalization. Then we get

$$W[J] = N \int [\mathcal{D}C] \Delta_f(C) \cdot [\prod_{\beta,z} \delta(f^\beta(C(z)) - \varphi^\beta(z))] \cdot \exp\{i \int d^4x (\mathcal{L} + J_\alpha^\mu C_\mu^\alpha)\}. \quad (10.19)$$

In fact, we can use this formula as our start-point of path integral quantization of gravitational gauge field, so we need not be worried about the influences of the third lines of eq.(10.18).

Use another functional

$$\exp\left\{-\frac{i}{2\alpha} \int d^4x \eta_{\alpha\beta} \varphi^\alpha(x) \varphi^\beta(x)\right\}, \quad (10.20)$$

times both sides of eq.(10.19) and then make functional integration $\int [\mathcal{D}\varphi]$, we get

$$W[J] = N \int [\mathcal{D}C] \Delta_f(C) \cdot \exp\left\{i \int d^4x (\mathcal{L} - \frac{1}{2\alpha} \eta_{\alpha\beta} f^\alpha f^\beta + J_\alpha^\mu C_\mu^\alpha)\right\}. \quad (10.21)$$

Now, let's discuss the contribution from $\Delta_f(C)$ which is related to the ghost fields. Suppose that $g = \hat{U}_\epsilon$ is an infinitesimal gravitational gauge transformation. Then eq.(4.12) gives out

$${}^g C_\mu^\alpha(x) = C_\mu^\alpha(x) - \frac{1}{g} \mathbf{D}_{\mu\sigma}^\alpha \epsilon^\sigma, \quad (10.22)$$

where

$$\mathbf{D}_{\mu\sigma}^\alpha = \delta_\sigma^\alpha \partial_\mu - g \delta_\sigma^\alpha C_\mu^\beta \partial_\beta + g \partial_\sigma C_\mu^\alpha. \quad (10.23)$$

In order to deduce eq.(10.22), the following relation is used

$$\Lambda_\beta^\alpha = \delta_\beta^\alpha + \partial_\beta \epsilon^\alpha + o(\epsilon^2). \quad (10.24)$$

\mathbf{D}_μ can be regarded as the covariant derivative in adjoint representation, for

$$\mathbf{D}_\mu \epsilon = [D_\mu, \epsilon], \quad (10.25)$$

$$(\mathbf{D}_\mu \epsilon)^\alpha = \mathbf{D}_{\mu\sigma}^\alpha \epsilon^\sigma. \quad (10.26)$$

Using all these relations, we have,

$$f^\alpha({}^g C(x)) = f^\alpha(C) - \frac{1}{g} \int d^4y \frac{\delta f^\alpha(C(x))}{\delta C_\mu^\beta(y)} \mathbf{D}_{\mu\sigma}^\beta(y) \epsilon^\sigma(y) + o(\epsilon^2). \quad (10.27)$$

Therefore, according to eq.(10.7) and eq.(10.6), we get

$$\Delta_f^{-1}(C) = \int [\mathcal{D}\epsilon] \prod_{x,\alpha} \delta \left(-\frac{1}{g} \int d^4y \frac{\delta f^\alpha(C(x))}{\delta C_\mu^\beta(y)} \mathbf{D}_{\mu\sigma}^\beta(y) \epsilon^\sigma(y) \right). \quad (10.28)$$

Define

$$\begin{aligned} \mathbf{M}_\sigma^\alpha(x, y) &= -g \frac{\delta}{\delta \epsilon^\sigma(y)} f^\alpha(gC(x)) \\ &= \int d^4z \frac{\delta f^\alpha(C(x))}{\delta C_\mu^\beta(z)} \mathbf{D}_{\mu\sigma}^\beta(z) \delta(z - y). \end{aligned} \quad (10.29)$$

Then eq.(10.28) is changed into

$$\begin{aligned} \Delta_f^{-1}(C) &= \int [\mathcal{D}\epsilon] \prod_{x,\alpha} \delta \left(-\frac{1}{g} \int d^4y \mathbf{M}_\sigma^\alpha(x, y) \epsilon^\sigma(y) \right) \\ &= \text{const.} \times (\det \mathbf{M})^{-1}. \end{aligned} \quad (10.30)$$

Therefore,

$$\Delta_f(C) = \text{const.} \times \det \mathbf{M}. \quad (10.31)$$

Put this constant into normalization constant, then generating functional eq.(10.21) is changed into

$$W[J] = N \int [\mathcal{D}C] \det \mathbf{M} \cdot \exp \left\{ i \int d^4x (\mathcal{L} - \frac{1}{2\alpha} \eta_{\alpha\beta} f^\alpha f^\beta + J_\alpha^\mu C_\mu^\alpha) \right\}. \quad (10.32)$$

In order to evaluate the contribution from $\det \mathbf{M}$, we introduce ghost fields $\eta^\alpha(x)$ and $\bar{\eta}_\alpha(x)$. Using the following relation

$$\int [\mathcal{D}\eta][\mathcal{D}\bar{\eta}] \exp \left\{ i \int d^4x d^4y \bar{\eta}_\alpha(x) \mathbf{M}_{\beta\alpha}^\alpha(x, y) \eta^\beta(y) \right\} = \text{const.} \times \det \mathbf{M} \quad (10.33)$$

and put the constant into the normalization constant, we can get

$$W[J] = N \int [\mathcal{D}C][\mathcal{D}\eta][\mathcal{D}\bar{\eta}] \exp \left\{ i \int d^4x (\mathcal{L} - \frac{1}{2\alpha} \eta_{\alpha\beta} f^\alpha f^\beta + \bar{\eta} \mathbf{M} \eta + J_\alpha^\mu C_\mu^\alpha) \right\}, \quad (10.34)$$

where $\int d^4x \bar{\eta} \mathbf{M} \eta$ is a simplified notation, whose explicit expression is

$$\int d^4x \bar{\eta} \mathbf{M} \eta = \int d^4x d^4y \bar{\eta}_\alpha(x) \mathbf{M}_{\beta\alpha}^\alpha(x, y) \eta^\beta(y). \quad (10.35)$$

The appearance of the non-trivial ghost fields is a inevitable result of the non-Abelian nature of the gravitational gauge group.

Now, let's take Lorentz covariant gauge condition,

$$f^\alpha(C) = \partial^\mu C_\mu^\alpha. \quad (10.36)$$

Then

$$\int d^4x \bar{\eta} \mathbf{M} \eta = - \int d^4x (\partial^\mu \bar{\eta}_\alpha(x)) \mathbf{D}_\mu^\alpha{}_\beta(x) \eta^\beta(x). \quad (10.37)$$

And eq.(10.34) is changed into

$$\begin{aligned} W[J, \beta, \bar{\beta}] = N \int [\mathcal{D}C][\mathcal{D}\eta][\mathcal{D}\bar{\eta}] \exp \left\{ i \int d^4x (\mathcal{L} - \frac{1}{2\alpha} \eta_{\alpha\beta} f^\alpha f^\beta \right. \\ \left. - (\partial^\mu \bar{\eta}_\alpha) \mathbf{D}_\mu^\alpha{}_\sigma \eta^\sigma + J_\alpha^\mu C_\mu^\alpha + \bar{\eta}_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta^\alpha \right\}, \end{aligned} \quad (10.38)$$

where we have introduced external sources $\eta^\alpha(x)$ and $\bar{\eta}_\alpha(x)$ of ghost fields.

The effective Lagrangian \mathcal{L}_{eff} is defined by

$$\mathcal{L}_{eff} \equiv \mathcal{L} - \frac{1}{2\alpha} \eta_{\alpha\beta} f^\alpha f^\beta - (\partial^\mu \bar{\eta}_\alpha) \mathbf{D}_\mu^\alpha{}_\sigma \eta^\sigma. \quad (10.39)$$

\mathcal{L}_{eff} can separate into free Lagrangian \mathcal{L}_F and interaction Lagrangian \mathcal{L}_I ,

$$\mathcal{L}_{eff} = \mathcal{L}_F + \mathcal{L}_I, \quad (10.40)$$

where

$$\begin{aligned} \mathcal{L}_F = & -\frac{1}{2} \eta^{\mu\rho} \eta^{\nu\sigma} \eta_{2\alpha\beta} \left[(\partial_\mu C_\nu^\alpha) (\partial_\rho C_\sigma^\beta) - (\partial_\mu C_\nu^\alpha) (\partial_\sigma C_\rho^\beta) \right] \\ & - \frac{1}{2\alpha} \eta_{2\alpha\beta} (\partial^\mu C_\mu^\alpha) (\partial^\nu C_\nu^\beta) - (\partial^\mu \bar{\eta}_\alpha) (\partial_\mu \eta^\alpha), \end{aligned} \quad (10.41)$$

$$\begin{aligned} \mathcal{L}_I = & -\frac{1}{2} (e^{I(C)} - 1) \eta^{\mu\rho} \eta^{\nu\sigma} \eta_{2\alpha\beta} \left[(\partial_\mu C_\nu^\alpha) (\partial_\rho C_\sigma^\beta) - (\partial_\mu C_\nu^\alpha) (\partial_\sigma C_\rho^\beta) \right] \\ & + g e^{I(C)} \eta^{\mu\rho} \eta^{\nu\sigma} \eta_{2\alpha\beta} (\partial_\mu C_\nu^\alpha - \partial_\nu C_\mu^\alpha) C_\rho^\delta \partial_\delta C_\sigma^\beta \\ & - \frac{1}{2} g^2 e^{I(C)} \eta^{\mu\rho} \eta^{\nu\sigma} \eta_{2\alpha\beta} (C_\mu^\delta \partial_\delta C_\nu^\alpha - C_\nu^\delta \partial_\delta C_\mu^\alpha) C_\rho^\epsilon \partial_\epsilon C_\sigma^\beta \\ & + g (\partial^\mu \bar{\eta}_\alpha) C_\mu^\beta (\partial_\beta \eta^\alpha) - g (\partial^\mu \bar{\eta}_\alpha) (\partial_\sigma C_\mu^\alpha) \eta^\sigma. \end{aligned} \quad (10.42)$$

From the interaction Lagrangian, we can see that ghost fields do not couple to $e^{I(C)}$. This is the reflection of the fact that ghost fields are not physical fields, they are virtual fields. Besides, the gauge fixing term does not couple to $e^{I(C)}$ either. Using effective Lagrangian \mathcal{L}_{eff} , the generating functional $W[J, \beta, \bar{\beta}]$ can be simplified to

$$W[J, \beta, \bar{\beta}] = N \int [\mathcal{D}C][\mathcal{D}\eta][\mathcal{D}\bar{\eta}] \exp \left\{ i \int d^4x (\mathcal{L}_{eff} + J_\alpha^\mu C_\mu^\alpha + \bar{\eta}_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta^\alpha) \right\}, \quad (10.43)$$

Use eq.(10.41), we can deduce propagator of gravitational gauge fields and ghost fields. First, we change its form to

$$\int d^4x \mathcal{L}_F = \int d^4x \left\{ \frac{1}{2} C_\nu^\alpha \left[\eta_{2\alpha\beta} \left(\eta^{\mu\nu} \partial^2 - \left(1 - \frac{1}{\alpha}\right) \partial^\mu \partial^\nu \right) \right] C_\nu^\beta + \bar{\eta}_\alpha \partial^2 \eta^\alpha \right\}. \quad (10.44)$$

Denote the propagator of gravitational gauge field as

$$-i \Delta_{F\mu\nu}^{\alpha\beta}(x), \quad (10.45)$$

and denote the propagator of ghost field as

$$-i \Delta_{F\beta}^\alpha(x). \quad (10.46)$$

They satisfy the following equation,

$$- \left[\eta_{2\alpha\beta} \left(\eta^{\mu\nu} \partial^2 - \left(1 - \frac{1}{\alpha}\right) \partial^\mu \partial^\nu \right) \right] \Delta_{F\nu\rho}^{\beta\gamma}(x) = \delta(x) \delta_\alpha^\gamma \delta_\rho^\mu, \quad (10.47)$$

$$-\partial^2 \Delta_{F\beta}^\alpha(x) = \delta_\beta^\alpha \delta(x). \quad (10.48)$$

Make Fourier transformations to momentum space

$$\Delta_{F\mu\nu}^{\alpha\beta}(x) = \int \frac{d^4k}{(2\pi)^4} \tilde{\Delta}_{F\mu\nu}^{\alpha\beta}(k) \cdot e^{ikx}, \quad (10.49)$$

$$\Delta_{F\beta}^\alpha(x) = \int \frac{d^4k}{(2\pi)^4} \tilde{\Delta}_{F\beta}^\alpha(k) \cdot e^{ikx}, \quad (10.50)$$

where $\tilde{\Delta}_{F\mu\nu}^{\alpha\beta}(k)$ and $\tilde{\Delta}_{F\beta}^\alpha(k)$ are corresponding propagators in momentum space. They satisfy the following equations,

$$\eta_{2\alpha\beta} \left[k^2 \eta^{\mu\nu} - \left(1 - \frac{1}{\alpha}\right) k^\mu k^\nu \right] \tilde{\Delta}_{F\nu\rho}^{\beta\gamma}(k) = \delta_\alpha^\gamma \delta_\rho^\mu, \quad (10.51)$$

$$k^2 \tilde{\Delta}_{F\beta}^\alpha(k) = \delta_\beta^\alpha. \quad (10.52)$$

The solutions to these two equations give out the propagators in momentum space,

$$-i \tilde{\Delta}_{F\mu\nu}^{\alpha\beta}(k) = \frac{-i}{k^2 - i\epsilon} \eta_2^{\alpha\beta} \left[\eta_{\mu\nu} - (1 - \alpha) \frac{k_\mu k_\nu}{k^2 - i\epsilon} \right], \quad (10.53)$$

$$-i \tilde{\Delta}_{F\beta}^\alpha(k) = \frac{-i}{k^2 - i\epsilon} \delta_\beta^\alpha. \quad (10.54)$$

It can be seen that the forms of these propagators are quite similar to those in traditional non-Abelian gauge theory. The only difference is that the metric is different.

The interaction Lagrangian \mathcal{L}_I is a function of gravitational gauge field C_μ^α and ghost fields η^α and $\bar{\eta}_\alpha$,

$$\mathcal{L}_I = \mathcal{L}_I(C, \eta, \bar{\eta}). \quad (10.55)$$

Then eq.(10.43) is changed into,

$$\begin{aligned} W[J, \beta, \bar{\beta}] &= N \int [\mathcal{D}C][\mathcal{D}\eta][\mathcal{D}\bar{\eta}] \exp \{i \int d^4x \mathcal{L}_I(C, \eta, \bar{\eta})\} \\ &\quad \cdot \exp \left\{ i \int d^4x (\mathcal{L}_F + J_\alpha^\mu C_\mu^\alpha + \bar{\eta}_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta^\alpha) \right\} \\ &= \exp \left\{ i \int d^4x \mathcal{L}_I \left(\frac{1}{i} \frac{\delta}{\delta J}, \frac{1}{i} \frac{\delta}{\delta \beta}, \frac{1}{-i} \frac{\delta}{\delta \bar{\beta}} \right) \right\} \cdot W_0[J, \beta, \bar{\beta}], \end{aligned} \quad (10.56)$$

where

$$\begin{aligned} W_0[J, \beta, \bar{\beta}] &= N \int [\mathcal{D}C][\mathcal{D}\eta][\mathcal{D}\bar{\eta}] \exp \left\{ i \int d^4x (\mathcal{L}_F + J_\alpha^\mu C_\mu^\alpha + \bar{\eta}_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta^\alpha) \right\} \\ &= \exp \left\{ \frac{i}{2} \int \int d^4x d^4y \left[J_\alpha^\mu(x) \Delta_{F\mu\nu}^{\alpha\beta}(x-y) J_\beta^\nu(y) \right. \right. \\ &\quad \left. \left. + \bar{\eta}_\alpha(x) \Delta_{F\beta}^\alpha(x-y) \eta^\beta(y) \right] \right\} \end{aligned} \quad (10.57)$$

Finally, let's discuss Feynman rules. Here, we only give out the lowest order interactions in gravitational gauge theory. It is known that, a vertex can involve arbitrary number of gravitational gauge fields. Therefore, it is impossible to list all Feynman rules for all kinds of vertex.

The interaction Lagrangian between gravitational gauge field and ghost field is

$$+g(\partial^\mu \bar{\eta}_\alpha) C_\mu^\beta (\partial^\beta \eta^\alpha) - g(\partial^\mu \bar{\eta}_\alpha) (\partial_\sigma C_\mu^\alpha) \eta^\sigma. \quad (10.58)$$

This vertex belongs to $C_\mu^\alpha(k) \bar{\eta}_\beta(q) \eta^\delta(p)$ three body interactions, its Feynman rule is

$$-ig\delta_\delta^\beta q^\mu p_\alpha + ig\delta_\alpha^\beta q^\mu k_\delta. \quad (10.59)$$

The lowest order interaction Lagrangian between gravitational gauge field and Dirac field is

$$g\bar{\psi}\gamma^\mu \partial_\alpha \psi C_\mu^\alpha - g\eta_{1\alpha}^\mu \bar{\psi}\gamma^\nu \partial_\nu \psi C_\mu^\alpha - gm\eta_{1\alpha}^\mu \bar{\psi}\psi C_\mu^\alpha. \quad (10.60)$$

This vertex belongs to $C_\mu^\alpha(k)\bar{\psi}(q)\psi(p)$ three body interactions, its Feynman rule is

$$-g\gamma^\mu p_\alpha + g\eta_{1\alpha}^\mu \gamma^\nu p_\nu - img\eta_{1\alpha}^\mu. \quad (10.61)$$

The lowest order interaction Lagrangian between gravitational gauge field and real scalar field is

$$g\eta^{\mu\nu}C_\mu^\alpha(\partial_\nu\phi)(\partial_\alpha\phi) - \frac{1}{2}g\eta_{1\alpha}^\mu C_\mu^\alpha((\partial^\nu\phi)(\partial_\nu\phi) + m^2\phi^2). \quad (10.62)$$

This vertex belongs to $C_\mu^\alpha(k)\phi(q)\phi(p)$ three body interactions, its Feynman rule is

$$-ig\eta^{\mu\nu}(p_\nu q_\alpha + q_\nu p_\alpha) - ig\eta_{1\alpha}^\mu(-p^\nu q_\nu + m^2). \quad (10.63)$$

The lowest order interaction Lagrangian between gravitational gauge field and complex scalar field is

$$g\eta^{\mu\nu}C_\mu^\alpha((\partial_\alpha\phi)(\partial_\nu\phi^*) + (\partial_\alpha\phi^*)(\partial_\nu\phi)) - g\eta_{1\alpha}^\mu C_\mu^\alpha((\partial^\nu\phi)(\partial_\nu\phi^*) + m^2\phi\phi^*). \quad (10.64)$$

This vertex belongs to $C_\mu^\alpha(k)\phi^*(-q)\phi(p)$ three body interactions, its Feynman rule is

$$ig\eta^{\mu\nu}(p_\nu q_\alpha + q_\nu p_\alpha) - ig\eta_{1\alpha}^\mu(p^\nu q_\nu + m^2). \quad (10.65)$$

The lowest order coupling between vector field and gravitational gauge field is

$$g\eta^{\mu\rho}\eta^{\nu\sigma}A_{0\mu\nu}C_\rho^\alpha\partial_\alpha A_\sigma \\ + (g\eta_{1\tau}^\lambda C_\lambda^\tau)(-\frac{1}{4}\eta^{\mu\rho}\eta^{\nu\sigma}A_{0\mu\nu}A_{0\rho\sigma} - \frac{m^2}{2}\eta^{\mu\nu}A_\mu A_\nu), \quad (10.66)$$

where

$$A_{0\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (10.67)$$

This vertex belongs to $C_\mu^\alpha(k)A_\rho(p)A_\sigma(q)$ three body interactions. Its Feynman rule is

$$-ig\eta^{\mu\beta}\eta^{\rho\sigma}(p_\beta q_\alpha + p_\alpha q_\beta) + ig\eta^{\mu\rho}\eta^{\sigma\beta}p_\beta q_\alpha + ig\eta^{\mu\sigma}\eta^{\rho\beta}q_\beta p_\alpha \\ + \frac{i}{2}g\eta_{1\alpha}^\mu\eta^{\lambda\beta}\eta^{\rho\sigma}(p_\lambda q_\beta + q_\lambda p_\beta) \\ - \frac{i}{2}g\eta_{1\alpha}^\mu\eta^{\rho\beta}\eta^{\nu\sigma}p_\nu q_\beta - \frac{i}{2}g\eta_{1\alpha}^\mu\eta^{\rho\nu}\eta^{\beta\sigma}q_\nu p_\beta - igm^2\eta_{1\alpha}^\mu\eta^{\rho\sigma}. \quad (10.68)$$

The lowest order self coupling of gravitational gauge fields is

$$g\eta^{\mu\rho}\eta^{\nu\sigma}\eta_{2\alpha\beta}[(\partial_\mu C_\nu^\alpha)C_\rho^{\beta_1}(\partial_{\beta_1}C_\sigma^\beta) - (\partial_\mu C_\nu^\alpha)C_\sigma^{\beta_1}(\partial_{\beta_1}C_\rho^\beta)] \\ - \frac{1}{4}(g\eta_{1\tau}^\lambda C_\lambda^\tau)\eta^{\mu\rho}\eta^{\nu\sigma}\eta_{2\alpha\beta}F_{0\mu\nu}^\alpha F_{0\rho\sigma}^\beta. \quad (10.69)$$

This vertex belongs to $C_\nu^\alpha(p)C_\sigma^\beta(q)C_\rho^\gamma(r)$ three body interactions. Its Feynman rule is

$$\begin{aligned}
& -ig[\eta^{\mu\rho}\eta^{\nu\sigma}\eta_{2\alpha\beta}(p_\mu q_\gamma + q_\mu p_\gamma) + \eta^{\mu\sigma}\eta^{\nu\rho}\eta_{2\alpha\gamma}(p_\mu r_\beta + r_\mu p_\beta) \\
& + \eta^{\mu\nu}\eta^{\rho\sigma}\eta_{2\gamma\beta}(q_\mu r_\alpha + r_\mu q_\alpha)] \\
& + ig[\eta^{\mu\sigma}\eta^{\nu\rho}\eta_{2\alpha\beta}p_\mu q_\gamma + \eta^{\mu\nu}\eta^{\rho\sigma}\eta_{2\alpha\beta}q_\mu p_\gamma + \eta^{\mu\rho}\eta^{\nu\sigma}\eta_{2\alpha\gamma}p_\mu r_\beta \\
& + \eta^{\mu\nu}\eta^{\rho\sigma}\eta_{2\alpha\gamma}r_\mu p_\beta + \eta^{\mu\rho}\eta^{\nu\sigma}\eta_{2\beta\gamma}q_\mu r_\alpha + \eta^{\mu\sigma}\eta^{\nu\rho}\eta_{2\beta\gamma}r_\mu q_\alpha] \\
& + ig\eta_{1\gamma}^\rho\eta_{2\alpha\beta}(p_\mu q^\mu\eta^{\nu\sigma} - p^\sigma q^\nu) + ig\eta_{1\alpha}^\nu\eta_{2\beta\gamma}(q_\mu r^\mu\eta^{\rho\sigma} - r^\sigma q^\rho) \\
& + ig\eta_{1\beta}^\sigma\eta_{2\alpha\gamma}(r_\mu p^\mu\eta^{\rho\nu} - p^\rho r^\nu)
\end{aligned} \tag{10.70}$$

It could be found that all Feynman rules for vertex is proportional to energy-momenta of one or more particles, which is one of the most important properties of gravitational interactions. In fact, this interaction property is expected for gravitational interactions, for energy-momentum is the source of gravity.

11 Renormalization

In gravitational gauge theory, the gravitational coupling constant has the dimensionality of negative powers of mass. According to traditional theory of power counting law, it seems that the gravitational gauge theory is a kind of non-renormalizable theory. But this result is not correct. The power counting law does not work here. General speaking, power counting law does not work when a theory has gauge symmetry. If a theory has gauge symmetry, the constraints from gauge symmetry will make some divergence cancel each other. In gravitational gauge theory, this mechanism works very well. In this chapter, we will give a strict formal proof on the renormalization of the gravitational gauge theory. We will find that the effect of renormalization is just a scale transformation of the original theory. Though there are infinite number of divergent vertexes in the gravitational gauge theory, we need not introduce infinite number of interaction terms that do not exist in the original Lagrangian and infinite number of parameters. All the divergent vertex can find its correspondence in the original Lagrangian. Therefore, in renormalization, what we need to do is not to introduce extra interaction terms to cancel divergent terms, but to redefine the fields, coupling constants and some other parameters of the original theory. Because most of counterterms come from the factor $e^{I(C)}$, this factor is key important for renormalization. Without this factor, the theory is

non-renormalizable. In a word, the gravitational gauge theory is a renormalizable gauge theory. Now, let's start our discussion on renormalization from the generalized BRST transformations. Our proof is quite similar to the proof of the renormalizability of non-Abelian gauge field theory.[29, 30, 31, 32, 33, 34]

The generalized BRST transformations are

$$\delta C_\mu^\alpha = -\mathbf{D}_\mu^\alpha{}_\beta \eta^\beta \delta\lambda, \quad (11.1)$$

$$\delta \eta^\alpha = g \eta^\sigma (\partial_\sigma \eta^\alpha) \delta\lambda, \quad (11.2)$$

$$\delta \bar{\eta}_\alpha = \frac{1}{\alpha} \eta_{\alpha\beta} f^\beta \delta\lambda, \quad (11.3)$$

$$\delta \eta^{\mu\nu} = 0, \quad (11.4)$$

$$\delta \eta_{1\alpha}^\mu = -g \eta_{1\sigma}^\mu (\partial_\alpha \eta^\sigma) \delta\lambda, \quad (11.5)$$

$$\delta \eta_{2\alpha\beta} = -g (\eta_{2\alpha\sigma} (\partial_\beta \eta^\sigma) + \eta_{2\sigma\beta} (\partial_\alpha \eta^\sigma)) \delta\lambda, \quad (11.6)$$

where $\delta\lambda$ is an infinitesimal Grassman constant. It can be strictly proved that the generalized BRST transformations for fields C_μ^α and η^α are nilpotent:

$$\delta(\mathbf{D}_\mu^\alpha{}_\beta \eta^\beta) = 0, \quad (11.7)$$

$$\delta(\eta^\sigma (\partial_\sigma \eta^\alpha)) = 0. \quad (11.8)$$

It means that all second order variations of fields vanish.

Using the above transformation rules, it can be strictly proved that the generalized BRST transformation for gauge field strength tensor $F_{\mu\nu}^\alpha$ is

$$\delta F_{\mu\nu}^\alpha = -g \left(-(\partial_\sigma \eta^\alpha) F_{\mu\nu}^\sigma + \eta^\sigma (\partial_\sigma F_{\mu\nu}^\alpha) \right) \delta\lambda, \quad (11.9)$$

and the transformation for the factor $e^{I(C)}$ is

$$\delta e^{I(C)} = -g \left((\partial_\alpha \eta^\alpha) e^{I(C)} + \eta^\alpha (\partial_\alpha e^{I(C)}) \right) \delta\lambda. \quad (11.10)$$

Therefore, under generalized BRST transformations, the Lagrangian \mathcal{L} given by eq.(10.1) transforms as

$$\delta \mathcal{L} = -g (\partial_\alpha (\eta^\alpha \mathcal{L})) \delta\lambda. \quad (11.11)$$

It is a total derivative term, its space-time integration vanishes, i.e., the action of eq.(10.2) is invariant under generalized BRST transformations,

$$\delta S = \delta \left(\int d^4x \mathcal{L} \right) = 0. \quad (11.12)$$

On the other hand, it can be strict proved that

$$\delta \left(-\frac{1}{2\alpha} \eta_{\alpha\beta} f^\alpha f^\beta + \bar{\eta}_\alpha \partial^\mu \mathbf{D}_{\mu\sigma}^\alpha \eta^\sigma \right) = 0. \quad (11.13)$$

The non-renormalized effective Lagrangian is denoted as $\mathcal{L}_{eff}^{[0]}$. It is given by

$$\mathcal{L}_{eff}^{[0]} = \mathcal{L} - \frac{1}{2\alpha} \eta_{\alpha\beta} f^\alpha f^\beta + \bar{\eta}_\alpha \partial^\mu \mathbf{D}_{\mu\sigma}^\alpha \eta^\sigma. \quad (11.14)$$

The effective action is defined by

$$S_{eff}^{[0]} = \int d^4x \mathcal{L}_{eff}^{[0]}. \quad (11.15)$$

Using eqs.(10.12-13), we can prove that this effective action is invariant under generalized BRST transformations,

$$\delta S_{eff}^{[0]} = 0. \quad (11.16)$$

This is a strict relation without any approximation. It is known that BRST symmetry plays a key role in the renormalization of gauge theory, for it ensures the validity of the Ward-Takahashi identities.

Before we go any further, we have to do another important work, i.e., to prove that the functional integration measure $[\mathcal{D}C][\mathcal{D}\eta][\mathcal{D}\bar{\eta}]$ is also generalized BRST invariant. We have said before that the functional integration measure $[\mathcal{D}C]$ is not a gauge invariant measure, therefore, it is highly important to prove that $[\mathcal{D}C][\mathcal{D}\eta][\mathcal{D}\bar{\eta}]$ is a generalized BRST invariant measure. BRST transformation is a kind of transformation which involves both bosonic fields and fermionic fields. For the sake of simplicity, let's formally denote all bosonic fields as $B = \{B_i\}$ and denote all fermionic fields as $F = \{F_i\}$. All fields that involve in generalized BRST transformation are simply denoted by (B, F) . Then, generalized BRST transformation is formally expressed as

$$(B, F) \rightarrow (B', F'). \quad (11.17)$$

The transformation matrix of this transformation is

$$J = \begin{pmatrix} \frac{\partial B_i}{\partial B'_j} & \frac{\partial B_i}{\partial F'_l} \\ \frac{\partial F_k}{\partial B'_j} & \frac{\partial F_k}{\partial F'_l} \end{pmatrix} = \begin{pmatrix} a & \alpha \\ \beta & b \end{pmatrix}, \quad (11.18)$$

where

$$a = \begin{pmatrix} \partial B_i \\ \partial B'_j \end{pmatrix}, \quad (11.19)$$

$$b = \left(\frac{\partial F_k}{\partial F'_l} \right), \quad (11.20)$$

$$\alpha = \left(\frac{\partial B_i}{\partial F'_l} \right), \quad (11.21)$$

$$\beta = \left(\frac{\partial F_k}{\partial B'_j} \right). \quad (11.22)$$

Matrixes a and b are bosonic square matrix while α and β generally are not square matrix. In order to calculate the Jacobian $\det(J)$. we realize the transformation (11.17) in two steps. The first step is a bosonic transformation

$$(B, F) \rightarrow (B', F). \quad (11.23)$$

The transformation matrix of this transformation is denoted as J_1 ,

$$J_1 = \begin{pmatrix} a - \alpha b^{-1} \beta & \alpha b^{-1} \\ 0 & 1 \end{pmatrix}. \quad (11.24)$$

Its Jacobian is

$$\det J_1 = \det(a - \alpha b^{-1} \beta). \quad (11.25)$$

Therefore,

$$\int \prod_i dB_i \prod_k dF_k = \int \prod_i dB'_i \prod_k dF_k \cdot \det(a - \alpha b^{-1} \beta). \quad (11.26)$$

The second step is a fermionic transformation,

$$(B', F) \rightarrow (B', F'). \quad (11.27)$$

Its transformation matrix is denoted as J_2 ,

$$J_2 = \begin{pmatrix} 1 & 0 \\ \beta & b \end{pmatrix}. \quad (11.28)$$

Its Jacobian is the inverse of the determinant of the transformation matrix,

$$(\det J_2)^{-1} = (\det b)^{-1}. \quad (11.29)$$

Using this relation, eq.(11.26) is changed into

$$\int \prod_i dB_i \prod_k dF_k = \int \prod_i dB'_i \prod_k dF'_k \cdot \det(a - \alpha b^{-1} \beta) (\det b)^{-1}. \quad (11.30)$$

For generalized BRST transformation, all non-diagonal matrix elements are proportional to Grassman constant $\delta\lambda$. Non-diagonal matrix α and β contains only non-diagonal matrix elements, so,

$$\alpha b^{-1} \beta \propto (\delta\lambda)^2 = 0. \quad (11.31)$$

It means that

$$\int \prod_i dB_i \prod_k dF_k = \int \prod_i dB'_i \prod_k dF'_k \cdot \det(a) \cdot (\det b)^{-1}. \quad (11.32)$$

Generally speaking, C_μ^α and $\partial_\nu C_\mu^\alpha$ are independent degrees of freedom, so are η^α and $\partial_\nu \eta^\alpha$. Using eqs.(11.1-3), we obtain

$$\begin{aligned} (\det a^{-1}) &= \det [(\delta_\beta^\alpha + g(\partial_\beta \eta^\alpha) \delta \lambda) \delta_\nu^\mu] \\ &= \prod_{\mu, \alpha, x} [(\delta_\alpha^\alpha + g(\partial_\alpha \eta^\alpha) \delta \lambda) \delta_\nu^\mu] \\ &= \prod_x (1 + g(\partial_\alpha \eta^\alpha) \delta \lambda). \end{aligned} \quad (11.33)$$

$$\begin{aligned} (\det b^{-1}) &= \det (\delta_\beta^\alpha + g(\partial_\beta \eta^\alpha) \delta \lambda) \\ &= \prod_x (1 + g(\partial_\alpha \eta^\alpha) \delta \lambda). \end{aligned} \quad (11.34)$$

In the second line of eq.(11.33), there is no summation over the repeated α index. Using these two relations, we have

$$\det(a) \cdot (\det b)^{-1} = \prod_x \mathbf{1} = 1. \quad (11.35)$$

Therefore, under generalized BRST transformation, functional integration measure $[\mathcal{DC}][\mathcal{D}\eta][\mathcal{D}\bar{\eta}]$ is invariant,

$$[\mathcal{DC}][\mathcal{D}\eta][\mathcal{D}\bar{\eta}] = [\mathcal{DC}'][\mathcal{D}\eta'][\mathcal{D}\bar{\eta}']. \quad (11.36)$$

Though both $[\mathcal{DC}]$ and $[\mathcal{D}\eta]$ are not invariant under generalized BRST transformation, their product is invariant under generalized BRST transformation. This result is interesting and important.

The generating functional $W^{[0]}[J, \eta_1, \eta_2]$ is

$$W^{[0]}[J, \eta_1, \eta_2] = N \int [\mathcal{DC}][\mathcal{D}\eta][\mathcal{D}\bar{\eta}] \exp \left\{ i \int d^4x (\mathcal{L}_{eff}^{[0]} + J_\alpha^\mu C_\mu^\alpha) \right\}, \quad (10.37)$$

where η_1 and η_2 are two tensors which are used to construct lagrangian density of pure gravitational gauge field. Because

$$\int d\eta^\beta d\bar{\eta}^\sigma \cdot \bar{\eta}^\alpha \cdot f(\eta, \bar{\eta}) = 0, \quad (11.38)$$

where $f(\eta, \bar{\eta})$ is a bilinear function of η and $\bar{\eta}$, we have

$$\int [\mathcal{DC}][\mathcal{D}\eta][\mathcal{D}\bar{\eta}] \cdot \bar{\eta}_\alpha(x) \cdot \exp \left\{ i \int d^4y (\mathcal{L}_{eff}^{[0]}(y) + J_\alpha^\mu(y) C_\mu^\alpha(y)) \right\} = 0. \quad (11.39)$$

If all fields are the fields after generalized BRST transformation, eq.(11.39) still holds, i.e.

$$\int [\mathcal{D}C'] [\mathcal{D}\eta'] [\mathcal{D}\bar{\eta}'] \cdot \bar{\eta}'_\alpha(x) \cdot \exp \left\{ i \int d^4y (\mathcal{L}'_{eff}{}^{[0]}(y) + J^\mu_\alpha(y) C'_\mu{}^\alpha(y)) \right\} = 0, \quad (11.40)$$

where $\mathcal{L}'_{eff}{}^{[0]}$ is the effective Lagrangian after generalized BRST transformation. Both functional integration measure and effective action $\int d^4y \mathcal{L}'_{eff}{}^{[0]}(y)$ are generalized BRST invariant, so, using eqs.(11.1-3), we get

$$\begin{aligned} \int [\mathcal{D}C] [\mathcal{D}\eta] [\mathcal{D}\bar{\eta}] & \left[\frac{1}{\alpha} \eta_{\alpha\beta} f^\beta(C(x)) \delta\lambda - i \bar{\eta}_\alpha(x) \int d^4z (J^\mu_\beta(z) \mathbf{D}_{\mu\sigma}^\beta(z) \eta^\sigma(z) \delta\lambda) \right] \\ & \cdot \exp \left\{ i \int d^4y (\mathcal{L}_{eff}{}^{[0]}(y) + J^\mu_\alpha(y) C_\mu{}^\alpha(y)) \right\} = 0. \end{aligned} \quad (11.41)$$

This equation will lead to

$$\frac{1}{\alpha} \eta_{\alpha\beta} f^\beta \left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right) W^{[0]}[J, \eta_1, \eta_2] - \int d^4y J^\mu_\beta(y) \mathbf{D}_{\mu\sigma}^\beta \left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right) W^{[0]\sigma}_\alpha[y, x, J, \eta_1, \eta_2] = 0, \quad (11.42)$$

where

$$W^{[0]\sigma}_\alpha[y, x, J, \eta_1, \eta_2] = Ni \int [\mathcal{D}C] [\mathcal{D}\eta] [\mathcal{D}\bar{\eta}] \bar{\eta}_\alpha(x) \eta^\sigma(y) \exp \left\{ i \int d^4z (\mathcal{L}_{eff}{}^{[0]} + J^\mu_\alpha C_\mu{}^\alpha) \right\}. \quad (10.43)$$

This is the generalized Ward-Takahashi identity for generating functional $W^{[0]}[J]$.

Now, let's introduce the external sources of ghost fields, then the generation functional becomes

$$W^{[0]}[J, \beta, \bar{\beta}, \eta_1, \eta_2] = N \int [\mathcal{D}C] [\mathcal{D}\eta] [\mathcal{D}\bar{\eta}] \exp \left\{ i \int d^4x (\mathcal{L}_{eff}{}^{[0]} + J^\mu_\alpha C_\mu{}^\alpha + \bar{\eta}_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta^\alpha) \right\}, \quad (11.44)$$

In renormalization of the theory, we have to introduce external sources K^μ_α and L_α of the following composite operators,

$$\mathbf{D}_{\mu\beta}^\alpha \eta^\beta, \quad g \eta^\sigma (\partial_\sigma \eta^\alpha). \quad (11.45)$$

Then the effective Lagrangian becomes

$$\begin{aligned} \tilde{\mathcal{L}}^{[0]}(C, \eta, \bar{\eta}, K, L, \eta_1, \eta_2) &= \mathcal{L} - \frac{1}{2\alpha} \eta_{\alpha\beta} f^\alpha f^\beta + \bar{\eta}_\alpha \partial^\mu \mathbf{D}_{\mu\sigma}^\alpha \eta^\sigma + K^\mu_\alpha \mathbf{D}_{\mu\beta}^\alpha \eta^\beta + g L_\alpha \eta^\sigma (\partial_\sigma \eta^\alpha) \\ &= \mathcal{L}_{eff}{}^{[0]} + K^\mu_\alpha \mathbf{D}_{\mu\beta}^\alpha \eta^\beta + g L_\alpha \eta^\sigma (\partial_\sigma \eta^\alpha). \end{aligned} \quad (11.46)$$

Then,

$$\tilde{S}^{[0]} [C, \eta, \bar{\eta}, K, L, \eta_1, \eta_2] = \int d^4x \tilde{\mathcal{L}}^{[0]} (C, \eta, \bar{\eta}, K, L, \eta_1, \eta_2). \quad (11.47)$$

It is easy to deduce that

$$\frac{\delta \tilde{S}^{[0]}}{\delta K_\alpha^\mu} = \mathbf{D}_{\mu\beta}^\alpha \eta^\beta, \quad (11.48)$$

$$\frac{\delta \tilde{S}^{[0]}}{\delta L_\alpha} = g\eta^\sigma (\partial_\sigma \eta^\alpha). \quad (11.49)$$

The generating functional now becomes,

$$W^{[0]} [J, \beta, \bar{\beta}, K, L, \eta_1, \eta_2] = N \int [\mathcal{D}C][\mathcal{D}\eta][\mathcal{D}\bar{\eta}] \exp \left\{ i \int d^4x (\tilde{\mathcal{L}}^{[0]} + J_\alpha^\mu C_\mu^\alpha + \bar{\eta}_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta^\alpha) \right\}. \quad (11.50)$$

In previous discussion, we have already proved that $S_{eff}^{[0]}$ is generalized BRST invariant. External sources K_α^μ and L_α keep unchanged under generalized BRST transformation. Using nilpotent property of generalized BRST transformation, it is easy to prove that the two new terms $K_\alpha^\mu \mathbf{D}_{\mu\beta}^\alpha \eta^\beta$ and $gL_\alpha \eta^\sigma (\partial_\sigma \eta^\alpha)$ in $\tilde{\mathcal{L}}^{[0]}$ are also generalized BRST invariant,

$$\delta(K_\alpha^\mu \mathbf{D}_{\mu\beta}^\alpha \eta^\beta) = 0, \quad (11.51)$$

$$\delta(gL_\alpha \eta^\sigma (\partial_\sigma \eta^\alpha)) = 0. \quad (11.52)$$

Therefore, the action given by (11.47) are generalized BRST invariant,

$$\delta \tilde{S}^{[0]} = 0. \quad (11.53)$$

It gives out

$$\int d^4x \left\{ -(\mathbf{D}_{\mu\beta}^\alpha \eta^\beta(x)) \delta\lambda \frac{\delta}{\delta C_\mu^\alpha(x)} + g\eta^\sigma(x) (\partial_\sigma \eta^\alpha(x)) \delta\lambda \frac{\delta}{\delta \eta^\alpha(x)} \right. \quad (11.54)$$

$$\left. + \frac{1}{\alpha} \eta_{\alpha\beta} f^\beta(C(x)) \delta\lambda \frac{\delta}{\delta \bar{\eta}_\alpha(x)} - L_{1\alpha}^\mu \delta\lambda \frac{\delta}{\delta \eta_{1\alpha}^\mu} - L_{2\alpha\beta} \delta\lambda \frac{\delta}{\delta \eta_{2\alpha\beta}} \right\} \tilde{S}^{[0]} = 0,$$

where

$$L_{1\alpha}^\mu = g\eta_{1\sigma}^\mu (\partial_\alpha \eta^\sigma), \quad (11.55)$$

$$L_{2\alpha\beta} = g[\eta_{2\alpha\sigma} (\partial_\beta \eta^\sigma) + \eta_{2\sigma\beta} (\partial_\beta \eta^\sigma)]. \quad (11.56)$$

Using relations (11.48-49), we can get

$$\int d^4x \left\{ \frac{\delta \tilde{S}^{[0]}}{\delta K_\alpha^\mu(x)} \frac{\delta \tilde{S}^{[0]}}{\delta C_\mu^\alpha(x)} + \frac{\delta \tilde{S}^{[0]}}{\delta L_\alpha(x)} \frac{\delta \tilde{S}^{[0]}}{\delta \eta^\alpha(x)} \right. \quad (11.57)$$

$$\left. + \frac{1}{\alpha} \eta_{\alpha\beta} f^\beta(C(x)) \frac{\delta \tilde{S}^{[0]}}{\delta \bar{\eta}_\alpha(x)} + L_{1\alpha}^\mu(x) \frac{\delta \tilde{S}^{[0]}}{\delta \eta_{1\alpha}^\mu(x)} + L_{2\alpha\beta}(x) \frac{\delta \tilde{S}^{[0]}}{\delta \eta_{2\alpha\beta}(x)} \right\} = 0.$$

On the other hand, from (11.46-47), we can obtain that

$$\frac{\delta \tilde{S}^{[0]}}{\delta \bar{\eta}_\alpha(x)} = \partial^\mu \left(\mathbf{D}_{\mu\beta}^\alpha \eta^\beta(x) \right). \quad (11.58)$$

Combine (11.48) with (11.58), we get

$$\frac{\delta \tilde{S}^{[0]}}{\delta \bar{\eta}_\alpha(x)} = \partial^\mu \left(\frac{\delta \tilde{S}^{[0]}}{\delta K_\alpha^\mu(x)} \right). \quad (11.59)$$

In generation functional $W^{[0]}[J, \beta, \bar{\beta}, K, L, \eta_1, \eta_2]$, all fields are integrated, so, if we set all fields to the fields after generalized BRST transformations, the final result should not be changed, i.e.

$$\begin{aligned} \tilde{W}^{[0]} [J, \beta, \bar{\beta}, K, L, \eta_1, \eta_2] &= N \int [\mathcal{D}C'] [\mathcal{D}\eta'] [\mathcal{D}\bar{\eta}] \\ &\cdot \exp \left\{ i \int d^4x (\mathcal{L}^{[0]}(C', \eta', \bar{\eta}', K, L, \eta_1, \eta_2) + J_\alpha^\mu C_\mu^\alpha + \bar{\eta}'_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta'^\alpha) \right\}. \end{aligned} \quad (11.60)$$

Both action (11.47) and functional integration measure $[\mathcal{D}C][\mathcal{D}\eta][\mathcal{D}\bar{\eta}]$ are generalized BRST invariant, so, the above relation gives out

$$\begin{aligned} &\int [\mathcal{D}C][\mathcal{D}\eta][\mathcal{D}\bar{\eta}] \left\{ i \int d^4x \left(J_\alpha^\mu \frac{\delta \tilde{S}^{[0]}}{\delta K_\alpha^\mu(x)} - \bar{\beta}_\alpha \frac{\delta \tilde{S}^{[0]}}{\delta L_\alpha(x)} \right. \right. \\ &\left. \left. + \frac{1}{\alpha} \eta_{\alpha\sigma} f^\alpha \beta^\sigma - L_{1\alpha}^\mu(x) \frac{\delta \tilde{S}^{[0]}}{\delta \eta_{1\alpha}^\mu(x)} - L_{2\alpha\beta}(x) \frac{\delta \tilde{S}^{[0]}}{\delta \eta_{2\alpha\beta}(x)} \right) \right\} \\ &\cdot \exp \left\{ i \int d^4y (\tilde{\mathcal{L}}^{[0]}(C, \eta, \bar{\eta}, K, L, \eta_1, \eta_2) + J_\alpha^\mu C_\mu^\alpha + \bar{\eta}_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta^\alpha) \right\} = 0. \end{aligned} \quad (11.61)$$

In order to obtain the above relation, the following relation is used,

$$\begin{aligned} &\int d^4x \tilde{\mathcal{L}}^{[0]}(C', \eta', \bar{\eta}', K, L, \eta_1, \eta_2) \\ &= \int d^4x \left[\tilde{\mathcal{L}}^{[0]}(C, \eta, \bar{\eta}, K, L, \eta_1, \eta_2) - \delta \eta_{1\alpha}^\mu \frac{\delta \tilde{S}^{[0]}}{\delta \eta_{1\alpha}^\mu(x)} - \delta \eta_{2\alpha\beta} \frac{\delta \tilde{S}^{[0]}}{\delta \eta_{2\alpha\beta}(x)} \right]. \end{aligned} \quad (11.62)$$

On the other hand, because the ghost field $\bar{\eta}_\alpha$ was integrated in functional $W^{[0]}[J, \beta, \bar{\beta}, K, L, \eta_1, \eta_2]$, if we use $\bar{\eta}'_\alpha$ in the in functional integration, it will not

change the generating functional. That is

$$\begin{aligned} \tilde{W}^{[0]} [J, \beta, \bar{\beta}, K, L, \eta_1, \eta_2] &= N \int [\mathcal{DC}] [\mathcal{D}\eta] [\mathcal{D}\bar{\eta}'] \\ &\cdot \exp \left\{ i \int d^4x \left(\tilde{\mathcal{L}}^{[0]} (C, \eta, \bar{\eta}', K, L, \eta_1, \eta_2) + J_\alpha^\mu C_\mu^\alpha + \bar{\eta}'_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta^\alpha \right) \right\}. \end{aligned} \quad (11.63)$$

Suppose that

$$\bar{\eta}'_\alpha = \bar{\eta}_\alpha + \delta \bar{\eta}_\alpha. \quad (11.64)$$

Then (11.63) and (11.50) will gives out

$$\begin{aligned} &\int [\mathcal{DC}] [\mathcal{D}\eta] [\mathcal{D}\bar{\eta}] \left\{ \int d^4x \delta \bar{\eta}_\alpha \left(\frac{\delta \tilde{S}^{[0]}}{\delta \bar{\eta}_\alpha(x)} + \beta^\alpha(x) \right) \right\} \\ &\cdot \exp \left\{ i \int d^4y \left(\tilde{\mathcal{L}}^{[0]} (C, \eta, \bar{\eta}, K, L, \eta_1, \eta_2) + J_\alpha^\mu C_\mu^\alpha + \bar{\eta}_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta^\alpha \right) \right\} = 0. \end{aligned} \quad (11.65)$$

Because $\delta \bar{\eta}_\alpha$ is an arbitrary variation, from (11.65), we will get

$$\begin{aligned} &\int [\mathcal{DC}] [\mathcal{D}\eta] [\mathcal{D}\bar{\eta}] \left(\frac{\delta \tilde{S}^{[0]}}{\delta \bar{\eta}_\alpha(x)} + \beta^\alpha(x) \right) \\ &\cdot \exp \left\{ i \int d^4y \left(\tilde{\mathcal{L}}^{[0]} (C, \eta, \bar{\eta}, K, L, \eta_1, \eta_2) + J_\alpha^\mu C_\mu^\alpha + \bar{\eta}_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta^\alpha \right) \right\} = 0. \end{aligned} \quad (11.66)$$

The generating functional of connected Green function is given by

$$\tilde{Z}^{[0]} [J, \beta, \bar{\beta}, K, L, \eta_1, \eta_2] = -i \ln \tilde{W}^{[0]} [J, \beta, \bar{\beta}, K, L, \eta_1, \eta_2]. \quad (11.67)$$

After Legendre transformation, we will get the generating functional of irreducible vertex $\tilde{\Gamma}^{[0]} [C, \bar{\eta}, \eta, K, L, \eta_1, \eta_2]$,

$$\begin{aligned} \tilde{\Gamma}^{[0]} [C, \bar{\eta}, \eta, K, L, \eta_1, \eta_2] &= \tilde{Z}^{[0]} [J, \beta, \bar{\beta}, K, L, \eta_1, \eta_2] \\ &\quad - \int d^4x \left(J_\alpha^\mu C_\mu^\alpha + \bar{\eta}_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta^\alpha \right). \end{aligned} \quad (11.68)$$

Functional derivative of the generating functional $\tilde{Z}^{[0]}$ gives out the classical fields C_μ^α , η^α and $\bar{\eta}_\alpha$,

$$C_\mu^\alpha = \frac{\delta \tilde{Z}^{[0]}}{\delta J_\alpha^\mu}, \quad (11.69)$$

$$\eta^\alpha = \frac{\delta \tilde{Z}^{[0]}}{\delta \bar{\beta}_\alpha}, \quad (11.70)$$

$$\bar{\eta}_\alpha = -\frac{\delta \tilde{Z}^{[0]}}{\delta \beta^\alpha}. \quad (11.71)$$

Then, functional derivative of the generating functional $\tilde{\Gamma}^{[0]}$ gives out external sources J_α^μ , $\bar{\beta}_\alpha$ and β^α ,

$$\frac{\delta \tilde{\Gamma}^{[0]}}{\delta C_\mu^\alpha} = -J_\alpha^\mu, \quad (11.72)$$

$$\frac{\delta \tilde{\Gamma}^{[0]}}{\delta \eta^\alpha} = \bar{\beta}_\alpha, \quad (11.73)$$

$$\frac{\delta \tilde{\Gamma}^{[0]}}{\delta \bar{\eta}_\alpha} = -\beta^\alpha. \quad (11.74)$$

Besides, there are two other relations which can be strictly deduced from (11.68),

$$\frac{\delta \tilde{\Gamma}^{[0]}}{\delta K_\alpha^\mu} = \frac{\delta \tilde{Z}^{[0]}}{\delta K_\alpha^\mu}, \quad \frac{\delta \tilde{\Gamma}^{[0]}}{\delta L_\alpha} = \frac{\delta \tilde{Z}^{[0]}}{\delta L_\alpha}. \quad (11.75)$$

$$\frac{\delta \tilde{\Gamma}^{[0]}}{\delta \eta_{1\alpha}^\mu} = \frac{\delta \tilde{Z}^{[0]}}{\delta \eta_{1\alpha}^\mu}, \quad \frac{\delta \tilde{\Gamma}^{[0]}}{\delta \eta_{2\alpha\beta}} = \frac{\delta \tilde{Z}^{[0]}}{\delta \eta_{2\alpha\beta}}. \quad (11.76)$$

It is easy to prove that

$$\begin{aligned} & i \frac{\delta \tilde{S}^{[0]}}{\delta K_\alpha^\mu(x)} \exp \left\{ i \int d^4y (\tilde{\mathcal{L}}^{[0]}(C, \eta, \bar{\eta}, K, L, \eta_1, \eta_2) + J_\alpha^\mu C_\mu^\alpha + \bar{\eta}_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta^\alpha) \right\} \\ &= \frac{\delta}{\delta K_\alpha^\mu(x)} \exp \left\{ i \int d^4y (\tilde{\mathcal{L}}^{[0]}(C, \eta, \bar{\eta}, K, L, \eta_1, \eta_2) + J_\alpha^\mu C_\mu^\alpha + \bar{\eta}_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta^\alpha) \right\}, \end{aligned} \quad (11.77)$$

$$\begin{aligned} & i \frac{\delta \tilde{S}^{[0]}}{\delta L_\alpha(x)} \exp \left\{ i \int d^4y (\tilde{\mathcal{L}}^{[0]}(C, \eta, \bar{\eta}, K, L, \eta_1, \eta_2) + J_\alpha^\mu C_\mu^\alpha + \bar{\eta}_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta^\alpha) \right\} \\ &= \frac{\delta}{\delta L_\alpha(x)} \exp \left\{ i \int d^4y (\tilde{\mathcal{L}}^{[0]}(C, \eta, \bar{\eta}, K, L, \eta_1, \eta_2) + J_\alpha^\mu C_\mu^\alpha + \bar{\eta}_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta^\alpha) \right\}. \end{aligned} \quad (11.78)$$

Use these two relations, we can change (11.61) into

$$\begin{aligned}
& \int [\mathcal{D}C][\mathcal{D}\eta][\mathcal{D}\bar{\eta}] \left\{ \int d^4x \left(J_\alpha^\mu(x) \frac{\delta}{\delta K_\alpha^\mu(x)} - \bar{\beta}_\alpha(x) \frac{\delta}{\delta L_\alpha(x)} \right. \right. \\
& \left. \left. + \frac{i}{\alpha} \eta_{\alpha\sigma} f^\alpha \left(\frac{1}{i} \frac{\delta}{\delta J_\gamma^\mu(x)} \right) \beta^\sigma(x) - L_{1\alpha}^\mu \left(\frac{1}{i} \frac{\delta}{\delta \beta} \right) \frac{\delta}{\delta \eta_{1\alpha}^\mu} - L_{2\alpha\beta} \left(\frac{1}{i} \frac{\delta}{\delta \beta} \right) \frac{\delta}{\delta \eta_{2\alpha\beta}} \right\} \\
& \cdot \exp \left\{ i \int d^4y \left(\tilde{\mathcal{L}}^{[0]}(C, \eta, \bar{\eta}, K, L, \eta_1, \eta_2) + J_\alpha^\mu C_\mu^\alpha + \bar{\eta}_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta^\alpha \right) \right\} = 0.
\end{aligned} \tag{11.79}$$

Using the definition of generating functional eq.(11.63), we can obtain

$$\begin{aligned}
& \int d^4x \left[J_\alpha^\mu(x) \frac{\delta \tilde{W}^{[0]}}{\delta K_\alpha^\mu(x)} - \bar{\beta}_\alpha(x) \frac{\delta \tilde{W}^{[0]}}{\delta L_\alpha(x)} + \frac{i}{\alpha} \eta_{\alpha\sigma} f^\alpha \left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right) \beta^\sigma(x) \tilde{W}^{[0]} \right. \\
& \left. - L_{1\alpha}^\mu \left(\frac{1}{i} \frac{\delta}{\delta \beta(x)} \right) \frac{\delta \tilde{W}^{[0]}}{\delta \eta_{1\alpha}^\mu(x)} - L_{2\alpha\beta} \left(\frac{1}{i} \frac{\delta}{\delta \beta(x)} \right) \frac{\delta \tilde{W}^{[0]}}{\delta \eta_{2\alpha\beta}(x)} \right] = 0.
\end{aligned} \tag{11.80}$$

Using the following three relations,

$$f^\beta \left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right) \tilde{W}^{[0]} = f^\beta(C(x)) \tilde{W}^{[0]}, \tag{11.81}$$

$$L_{1\alpha}^\mu \left(\frac{1}{i} \frac{\delta}{\delta \beta(x)} \right) \frac{\delta \tilde{W}^{[0]}}{\delta \eta_{1\alpha}^\mu(x)} = L_{1\alpha}^\mu(\eta(x)) \frac{\delta \tilde{W}^{[0]}}{\delta \eta_{1\alpha}^\mu(x)}, \tag{11.82}$$

$$L_{2\alpha\beta} \left(\frac{1}{i} \frac{\delta}{\delta \beta(x)} \right) \frac{\delta \tilde{W}^{[0]}}{\delta \eta_{2\alpha\beta}(x)} = L_{2\alpha\beta}(\eta(x)) \frac{\delta \tilde{W}^{[0]}}{\delta \eta_{2\alpha\beta}(x)}, \tag{11.83}$$

eq.(11.80) can be changed into,

$$\begin{aligned}
& \int d^4x \left[J_\alpha^\mu(x) \frac{\delta \tilde{W}^{[0]}}{\delta K_\alpha^\mu(x)} - \bar{\beta}_\alpha(x) \frac{\delta \tilde{W}^{[0]}}{\delta L_\alpha(x)} + \frac{i}{\alpha} \eta_{\alpha\sigma} f^\alpha \beta^\sigma(x) \tilde{W}^{[0]} \right. \\
& \left. - L_{1\alpha}^\mu \frac{\delta \tilde{W}^{[0]}}{\delta \eta_{1\alpha}^\mu(x)} - L_{2\alpha\beta} \frac{\delta \tilde{W}^{[0]}}{\delta \eta_{2\alpha\beta}(x)} \right] = 0.
\end{aligned} \tag{11.84}$$

Using relations (11.72-74), we can rewrite this equation into

$$\begin{aligned}
& \int d^4x \left\{ \frac{\delta \tilde{W}^{[0]}}{\delta K_\alpha^\mu(x)} \frac{\delta \tilde{\Gamma}^{[0]}}{\delta C_\mu^\alpha(x)} + \frac{\delta \tilde{W}^{[0]}}{\delta L_\alpha(x)} \frac{\delta \tilde{\Gamma}^{[0]}}{\delta \eta^\alpha(x)} + \frac{i}{\alpha} \eta_{\alpha\sigma} f^\alpha \tilde{W}^{[0]} \frac{\delta \tilde{\Gamma}^{[0]}}{\delta \bar{\eta}_\sigma(x)} \right. \\
& \left. + L_{1\alpha}^\mu \frac{\delta \tilde{W}^{[0]}}{\delta \eta_{1\alpha}^\mu(x)} + L_{2\alpha\beta} \frac{\delta \tilde{W}^{[0]}}{\delta \eta_{2\alpha\beta}(x)} \right\} = 0.
\end{aligned} \tag{11.85}$$

Using (11.67), we can obtain that

$$\frac{\delta \tilde{W}^{[0]}}{\delta K_\alpha^\mu(x)} = i \frac{\delta \tilde{\Gamma}^{[0]}}{\delta K_\alpha^\mu(x)} \cdot \tilde{W}^{[0]}, \quad (11.86)$$

$$\frac{\delta \tilde{W}^{[0]}}{\delta L_\alpha(x)} = i \frac{\delta \tilde{\Gamma}^{[0]}}{\delta L_\alpha(x)} \cdot \tilde{W}^{[0]}, \quad (11.87)$$

$$\frac{\delta \tilde{W}^{[0]}}{\delta \eta_{1\alpha}^\mu(x)} = i \frac{\delta \tilde{\Gamma}^{[0]}}{\delta \eta_{1\alpha}^\mu(x)} \cdot \tilde{W}^{[0]}, \quad (11.88)$$

$$\frac{\delta \tilde{W}^{[0]}}{\delta \eta_{2\alpha\beta}(x)} = i \frac{\delta \tilde{\Gamma}^{[0]}}{\delta \eta_{2\alpha\beta}(x)} \cdot \tilde{W}^{[0]}. \quad (11.89)$$

Then (11.76) is changed into

$$\int d^4x \left\{ \frac{\delta \tilde{\Gamma}^{[0]}}{\delta K_\alpha^\mu(x)} \frac{\delta \tilde{\Gamma}^{[0]}}{\delta C_\mu^\alpha(x)} + \frac{\delta \tilde{\Gamma}^{[0]}}{\delta L_\alpha(x)} \frac{\delta \tilde{\Gamma}^{[0]}}{\delta \eta^\alpha(x)} + \frac{1}{\alpha} \eta_{\alpha\sigma} f^\alpha \frac{\delta \tilde{\Gamma}^{[0]}}{\delta \bar{\eta}_\sigma(x)} \right. \\ \left. + L_{1\alpha}^\mu(x) \frac{\delta \tilde{\Gamma}^{[0]}}{\delta \eta_{1\alpha}^\mu(x)} + L_{2\alpha\beta}(x) \frac{\delta \tilde{\Gamma}^{[0]}}{\delta \eta_{2\alpha\beta}(x)} \right\} = 0. \quad (11.90)$$

Using (11.59) and (11.77), (11.65) becomes

$$\int [\mathcal{D}C][\mathcal{D}\eta][\mathcal{D}\bar{\eta}] \left[-i\partial^\mu \frac{\delta}{\delta K_\alpha^\mu(x)} + \beta^\alpha(x) \right] \\ \cdot \exp \left\{ i \int d^4y (\mathcal{L}^{[0]}(C, \eta, \bar{\eta}, K, L, \eta_1, \eta_2) + J_\alpha^\mu C_\mu^\alpha + \bar{\eta}_\alpha \beta^\alpha + \bar{\beta}_\alpha \eta^\alpha) \right\} = 0. \quad (11.91)$$

In above equation, the factor $-i\partial^\mu \frac{\delta}{\delta K_\alpha^\mu(x)} + \beta^\alpha(x)$ can move out of functional integration, then (11.91) gives out

$$\partial^\mu \frac{\delta \tilde{\Gamma}^{[0]}}{\delta K_\alpha^\mu(x)} = \frac{\delta \tilde{\Gamma}^{[0]}}{\delta \bar{\eta}_\alpha(x)}. \quad (11.92)$$

In order to obtain this relation, (11.63), (11.86) and (11.74) are used.

Define

$$\bar{\Gamma}^{[0]}[C, \bar{\eta}, \eta, K, L, \eta_1, \eta_2] = \tilde{\Gamma}^{[0]}[C, \bar{\eta}, \eta, K, L, \eta_1, \eta_2] + \frac{1}{2\alpha} \int d^4x \eta_{\alpha\beta} f^\alpha f^\beta. \quad (11.93)$$

It is easy to prove that

$$\frac{\delta \bar{\Gamma}^{[0]}}{\delta K_\alpha^\mu(x)} = \frac{\delta \tilde{\Gamma}^{[0]}}{\delta K_\alpha^\mu(x)}, \quad (11.94)$$

$$\frac{\delta\bar{\Gamma}^{[0]}}{\delta L_\alpha(x)} = \frac{\delta\tilde{\Gamma}^{[0]}}{\delta L_\alpha(x)}, \quad (11.95)$$

$$\frac{\delta\bar{\Gamma}^{[0]}}{\delta\eta^\alpha(x)} = \frac{\delta\tilde{\Gamma}^{[0]}}{\delta\eta^\alpha(x)}, \quad (11.96)$$

$$\frac{\delta\bar{\Gamma}^{[0]}}{\delta\bar{\eta}_\alpha(x)} = \frac{\delta\tilde{\Gamma}^{[0]}}{\delta\bar{\eta}_\alpha(x)}, \quad (11.97)$$

$$\frac{\delta\bar{\Gamma}^{[0]}}{\delta C_\mu^\alpha(x)} = \frac{\delta\tilde{\Gamma}^{[0]}}{\delta C_\mu^\alpha(x)} - \frac{1}{\alpha}\eta_{\alpha\beta}\partial^\mu f^\beta, \quad (11.98)$$

$$\frac{\delta\bar{\Gamma}^{[0]}}{\delta\eta_{1\alpha}^\mu(x)} = \frac{\delta\tilde{\Gamma}^{[0]}}{\delta\eta_{1\alpha}^\mu(x)}, \quad (11.99)$$

$$\frac{\delta\bar{\Gamma}^{[0]}}{\delta\eta_{2\alpha\beta}(x)} = \frac{\delta\tilde{\Gamma}^{[0]}}{\delta\eta_{2\alpha\beta}(x)}. \quad (11.100)$$

Using these relations, (11.92) and (11.90) are changed into

$$\partial^\mu \frac{\delta\bar{\Gamma}^{[0]}}{\delta K_\alpha^\mu(x)} = \frac{\delta\bar{\Gamma}^{[0]}}{\delta\bar{\eta}_\alpha(x)}, \quad (11.101)$$

$$\int d^4x \left\{ \frac{\delta\bar{\Gamma}^{[0]}}{\delta K_\alpha^\mu(x)} \frac{\delta\bar{\Gamma}^{[0]}}{\delta C_\mu^\alpha(x)} + \frac{\delta\bar{\Gamma}^{[0]}}{\delta L_\alpha(x)} \frac{\delta\bar{\Gamma}^{[0]}}{\delta\eta^\alpha(x)} + L_{1\alpha}^\mu \frac{\delta\bar{\Gamma}^{[0]}}{\delta\eta_{1\alpha}^\mu(x)} + L_{2\alpha\beta} \frac{\delta\bar{\Gamma}^{[0]}}{\delta\eta_{2\alpha\beta}(x)} \right\} = 0. \quad (11.102)$$

Eqs.(11.101-102) are generalized Ward-Takahashi identities of generating functional of regular vertex. It is the foundations of the renormalization of the gravitational gauge theory.

Generating functional $\tilde{\Gamma}^{[0]}$ is the generating functional of regular vertex with external sources, which is constructed from the Lagrangian $\tilde{\mathcal{L}}_{eff}^{[0]}$. It is a functional of physical field, therefore, we can make a functional expansion

$$\begin{aligned} \tilde{\Gamma}^{[0]} &= \sum_n \int \frac{\delta^n \tilde{\Gamma}^{[0]}}{\delta C_{\mu_1}^{\alpha_1}(x_1) \cdots \delta C_{\mu_n}^{\alpha_n}(x_n)} \Big|_{C=\eta=\bar{\eta}=0} C_{\mu_1}^{\alpha_1}(x_1) \cdots C_{\mu_n}^{\alpha_n}(x_n) d^4x_1 \cdots d^4x_n \\ &+ \sum_n \int \frac{\delta^2}{\delta\bar{\eta}_{\beta_1}(y_1)\delta\eta^{\beta_2}(y_2)} \frac{\delta^n \tilde{\Gamma}^{[0]}}{\delta C_{\mu_1}^{\alpha_1}(x_1) \cdots \delta C_{\mu_n}^{\alpha_n}(x_n)} \Big|_{C=\eta=\bar{\eta}=0} \\ &\quad \cdot \bar{\eta}_{\beta_1}(y_1)\eta^{\beta_2}(y_2) C_{\mu_1}^{\alpha_1}(x_1) \cdots C_{\mu_n}^{\alpha_n}(x_n) d^4y_1 d^4y_2 d^4x_1 \cdots d^4x_n \\ &+ \cdots. \end{aligned} \quad (11.103)$$

In this functional expansion, the expansion coefficients are regular vertexes with external sources. Before renormalization, these coefficients contain divergences. If we calculate these divergences in the methods of dimensional regularization, the form of these divergence will not violate gauge symmetry of the theory[36, 37]. In other words, in the method of dimensional regularization, gravitational gauge symmetry is not violated and the generating functional of regular vertex satisfies Ward-Takahashi identities (11.88-89). In order to eliminate the ultraviolet divergences of the theory, we need to introduce counterterms into Lagrangian. All these counterterms are formally denoted by $\delta\mathcal{L}$. Then the renormalized Lagrangian is

$$\tilde{\mathcal{L}}_{eff} = \tilde{\mathcal{L}}_{eff}^{[0]} + \delta\mathcal{L}. \quad (11.104)$$

Because $\delta\mathcal{L}$ contains all counterterms, $\tilde{\mathcal{L}}_{eff}$ is the Lagrangian density after complete renormalization. The generating functional of regular vertex which is calculated from $\tilde{\mathcal{L}}_{eff}$ is denoted by $\tilde{\Gamma}$. The regular vertexes calculated from this generating functional $\tilde{\Gamma}$ contain no ultraviolet divergence anymore. Then let external sources K_α^μ and L_α vanish, we will get generating functional Γ of regular vertex without external sources,

$$\Gamma = \tilde{\Gamma} |_{K=L=0}. \quad (11.105)$$

The regular vertexes which are generated from Γ will contain no ultraviolet divergence either. Therefore, the S-matrix for all physical process are finite. For a renormalizable theory, the counterterm $\delta\mathcal{L}$ only contain finite unknown parameters which are needed to be determined by experiments. If conterterm $\delta\mathcal{L}$ contains infinite unknown parameters, the theory will lost its predictive power and it is conventionally regarded as a non-renormalizable theory. Now, the main task for us is to prove that the conterterm $\delta\mathcal{L}$ for the gravitational gauge theory only contains a few unknown parameters. If we do this, we will have proved that the gravitational gauge theory is renormalizable.

Now, we use inductive method to prove the renormalizability of the gravitational gauge theory. In the previous discussion, we have proved that the generating functional of regular vertex before renormalization satisfies Ward-Takahashi identities (11.88-89). The effective Lagrangian density that contains all counterterms which cancel all divergences of l -loops ($0 \leq l \leq L$) is denoted by $\tilde{\mathcal{L}}^{[L]}$. $\tilde{\Gamma}^{[L]}$ is the generating functional of regular vertex which is calculated from $\tilde{\mathcal{L}}^{[L]}$. The regular vertex which is generated by $\tilde{\Gamma}^{[L]}$ will contain no divergence if the number of the loops of the diagram is not greater than L . We have proved that the generating functional $\tilde{\Gamma}^{[L]}$ satisfies Ward-Takahashi identities if $L = 0$. Hypothesize that Ward-Takahashi

identities are also satisfied when $L = N$, that is

$$\partial^\mu \frac{\delta \bar{\Gamma}^{[N]}}{\delta K_\alpha^\mu(x)} = \frac{\delta \bar{\Gamma}^{[N]}}{\delta \bar{\eta}_\alpha(x)}, \quad (11.106)$$

$$\int d^4x \left\{ \frac{\delta \bar{\Gamma}^{[N]}}{\delta K_\alpha^\mu(x)} \frac{\delta \bar{\Gamma}^{[N]}}{\delta C_\mu^\alpha(x)} + \frac{\delta \bar{\Gamma}^{[N]}}{\delta L_\alpha(x)} \frac{\delta \bar{\Gamma}^{[N]}}{\delta \eta^\alpha(x)} + L_{1\alpha}^\mu \frac{\delta \bar{\Gamma}^{[N]}}{\delta \eta_{1\alpha}^\mu(x)} + L_{2\alpha\beta} \frac{\delta \bar{\Gamma}^{[N]}}{\delta \eta_{2\alpha\beta}(x)} \right\} = 0. \quad (11.107)$$

Our goal is to prove that Ward-Takahashi identities are also satisfied when $L = N+1$.

Now, let's introduce a special product which is defined by

$$A * B \equiv \int d^4x \left\{ \frac{\delta A}{\delta K_\alpha^\mu(x)} \frac{\delta B}{\delta C_\mu^\alpha(x)} + \frac{\delta A}{\delta L_\alpha(x)} \frac{\delta B}{\delta \eta^\alpha(x)} \right\}. \quad (11.108)$$

Then (11.107) will be simplified to

$$\bar{\Gamma}^{[N]} * \bar{\Gamma}^{[N]} + \int d^4x \left(L_{1\alpha}^\mu \frac{\delta \bar{\Gamma}^{[N]}}{\delta \eta_{1\alpha}^\mu(x)} + L_{2\alpha\beta} \frac{\delta \bar{\Gamma}^{[N]}}{\delta \eta_{2\alpha\beta}(x)} \right) = 0. \quad (11.109)$$

$\bar{\Gamma}^{[N]}$ contains all contributions from all possible diagrams with arbitrary loops. The contribution from l -loop diagram is proportional to \hbar^l . We can expand $\bar{\Gamma}^{[N]}$ as a power series of \hbar^l ,

$$\bar{\Gamma}^{[N]} = \sum_M \hbar^M \bar{\Gamma}_M^{[N]}, \quad (11.110)$$

where $\bar{\Gamma}_M^{[N]}$ is the contribution from all M -loop diagrams. According to our inductive hypothesis, all $\bar{\Gamma}_M^{[N]}$ are finite is $M \leq N$. Therefore, divergence first appear in $\bar{\Gamma}_{N+1}^{[N]}$. Substitute (11.110) into (11.109), we will get

$$\sum_{M,L} \hbar^{M+L} \bar{\Gamma}_M^{[N]} * \bar{\Gamma}_L^{[N]} + \sum_M \hbar^M \int d^4x \left(L_{1\alpha}^\mu \frac{\delta \bar{\Gamma}_M^{[N]}}{\delta \eta_{1\alpha}^\mu(x)} + L_{2\alpha\beta} \frac{\delta \bar{\Gamma}_M^{[N]}}{\delta \eta_{2\alpha\beta}(x)} \right) = 0. \quad (11.111)$$

The $(N+1)$ -loop contribution of (11.111) is

$$\sum_{M=0}^{N+1} \bar{\Gamma}_M^{[N]} * \bar{\Gamma}_{N-M+1}^{[N]} + \int d^4x \left(L_{1\alpha}^\mu \frac{\delta \bar{\Gamma}_{N+1}^{[N]}}{\delta \eta_{1\alpha}^\mu(x)} + L_{2\alpha\beta} \frac{\delta \bar{\Gamma}_{N+1}^{[N]}}{\delta \eta_{2\alpha\beta}(x)} \right) = 0. \quad (11.112)$$

$\bar{\Gamma}_{N+1}^{[N]}$ can be separated into two parts: finite part $\bar{\Gamma}_{N+1,F}^{[N]}$ and divergent part $\bar{\Gamma}_{N+1,div}^{[N]}$, that is

$$\bar{\Gamma}_{N+1}^{[N]} = \bar{\Gamma}_{N+1,F}^{[N]} + \bar{\Gamma}_{N+1,div}^{[N]}. \quad (11.113)$$

$\bar{\Gamma}_{N+1,div}^{[N]}$ is a divergent function of $(4-D)$ if we calculate loop diagrams in dimensional regularization. In other words, all terms in $\bar{\Gamma}_{N+1,div}^{[N]}$ are divergent terms when $(4-D)$

approaches zero. Substitute (11.113) into (11.112), if we only concern divergent terms, we will get

$$\bar{\Gamma}_{N+1,div}^{[N]} * \bar{\Gamma}_0^{[N]} + \bar{\Gamma}_0^{[N]} * \bar{\Gamma}_{N+1,div}^{[N]} + \int d^4x \left(L_{1\alpha}^\mu \frac{\delta \bar{\Gamma}_{N+1,div}^{[N]}}{\delta \eta_{1\alpha}^\mu(x)} + L_{2\alpha\beta} \frac{\delta \bar{\Gamma}_{N+1,div}^{[N]}}{\delta \eta_{2\alpha\beta}(x)} \right) = 0. \quad (11.114)$$

$\bar{\Gamma}_{N+1,F}^{[N]}$ has no contribution to the divergent part. Because $\bar{\Gamma}_0^{[N]}$ represents contribution from tree diagram and counterterm has no contribution to tree diagram, we have

$$\bar{\Gamma}_0^{[N]} = \bar{\Gamma}_0^{[0]}. \quad (11.115)$$

Denote

$$\bar{\Gamma}_0 = \bar{\Gamma}_0^{[N]} = \tilde{S}^{[0]} + \frac{1}{2\alpha} \int d^4x \eta_{\alpha\beta} f^\alpha f^\beta. \quad (11.116)$$

Then (11.114) is changed into

$$\bar{\Gamma}_{N+1,div}^{[N]} * \bar{\Gamma}_0 + \bar{\Gamma}_0 * \bar{\Gamma}_{N+1,div}^{[N]} + \int d^4x \left(L_{1\alpha}^\mu \frac{\delta \bar{\Gamma}_{N+1,div}^{[N]}}{\delta \eta_{1\alpha}^\mu(x)} + L_{2\alpha\beta} \frac{\delta \bar{\Gamma}_{N+1,div}^{[N]}}{\delta \eta_{2\alpha\beta}(x)} \right) = 0. \quad (11.117)$$

Substitute (11.110) into (11.106), we get

$$\partial^\mu \frac{\delta \bar{\Gamma}_{N+1}^{[N]}}{\delta K_\alpha^\mu(x)} = \frac{\delta \bar{\Gamma}_{N+1}^{[N]}}{\delta \bar{\eta}_\alpha(x)}. \quad (11.118)$$

The finite part $\bar{\Gamma}_{N+1,F}^{[N]}$ has no contribution to the divergent part, so we have

$$\partial^\mu \frac{\delta \bar{\Gamma}_{N+1,div}^{[N]}}{\delta K_\alpha^\mu(x)} = \frac{\delta \bar{\Gamma}_{N+1,div}^{[N]}}{\delta \bar{\eta}_\alpha(x)}. \quad (11.119)$$

The operator \hat{g} and \hat{g}_1 are defined by

$$\hat{g} \triangleq \int d^4x \left\{ \frac{\delta \bar{\Gamma}_0}{\delta C_\mu^\alpha(x)} \frac{\delta}{\delta K_\alpha^\mu(x)} + \frac{\delta \bar{\Gamma}_0}{\delta L_\alpha(x)} \frac{\delta}{\delta \eta^\alpha(x)} + \frac{\delta \bar{\Gamma}_0}{\delta K_\alpha^\mu(x)} \frac{\delta}{\delta C_\mu^\alpha(x)} \right. \\ \left. + \frac{\delta \bar{\Gamma}_0}{\delta \eta^\alpha(x)} \frac{\delta}{\delta L_\alpha(x)} + L_{1\alpha}^\mu \frac{\delta}{\delta \eta_{1\alpha}^\mu(x)} + L_{2\alpha\beta} \frac{\delta}{\delta \eta_{2\alpha\beta}(x)} \right\}. \quad (11.120)$$

$$\hat{g}_1 \triangleq \int d^4x \left\{ \frac{\delta \bar{\Gamma}_0}{\delta C_\mu^\alpha(x)} \frac{\delta}{\delta K_\alpha^\mu(x)} + \frac{\delta \bar{\Gamma}_0}{\delta L_\alpha(x)} \frac{\delta}{\delta \eta^\alpha(x)} \right. \\ \left. + \frac{\delta \bar{\Gamma}_0}{\delta K_\alpha^\mu(x)} \frac{\delta}{\delta C_\mu^\alpha(x)} + \frac{\delta \bar{\Gamma}_0}{\delta \eta^\alpha(x)} \frac{\delta}{\delta L_\alpha(x)} \right\}. \quad (11.121)$$

Using definition eq.(11.120), (11.117) simplifies to

$$\hat{g}\bar{\Gamma}_{N+1,div}^{[N]} = 0. \quad (11.122)$$

Operators \hat{g} and \hat{g}_1 are not nilpotent operators. It can be proved that

$$\begin{aligned} \hat{g}_1^2 &= \int d^4x \left\{ \left[\frac{\delta}{\delta C_\mu^\alpha(x)} (\bar{\Gamma}_0 * \bar{\Gamma}_0) \right] \frac{\delta}{\delta K_\alpha^\mu(x)} + \left[\frac{\delta}{\delta L_\alpha(x)} (\bar{\Gamma}_0 * \bar{\Gamma}_0) \right] \frac{\delta}{\delta \eta^\alpha(x)} \right\} \\ &\quad - \int d^4x \left\{ \left[\frac{\delta}{\delta K_\alpha^\mu(x)} (\bar{\Gamma}_0 * \bar{\Gamma}_0) \right] \frac{\delta}{\delta C_\mu^\alpha(x)} + \left[\frac{\delta}{\delta \eta^\alpha(x)} (\bar{\Gamma}_0 * \bar{\Gamma}_0) \right] \frac{\delta}{\delta L_\alpha(x)} \right\}, \end{aligned} \quad (11.123)$$

$$\begin{aligned} \hat{g}\hat{g}_1 &= \int d^4x d^4y L_{1\alpha}^\mu(x) \left(\frac{\delta \bar{\Gamma}_0}{\delta C_\mu^\alpha(y)} \frac{\delta}{\delta K_\alpha^\mu(y)} + \frac{\delta \bar{\Gamma}_0}{\delta K_\alpha^\mu(y)} \frac{\delta}{\delta C_\mu^\alpha(y)} \right. \\ &\quad \left. + \frac{\delta \bar{\Gamma}_0}{\delta L_\alpha(y)} \frac{\delta}{\delta \eta^\alpha(y)} + \frac{\delta \bar{\Gamma}_0}{\delta \eta^\alpha(y)} \frac{\delta}{\delta L_\alpha(y)} \right) \frac{\delta}{\delta \eta_{1\alpha}^\mu(x)} \\ &\quad + \int d^4x d^4y L_{2\alpha\beta}(x) \left(\frac{\delta \bar{\Gamma}_0}{\delta C_\mu^\alpha(y)} \frac{\delta}{\delta K_\alpha^\mu(y)} + \frac{\delta \bar{\Gamma}_0}{\delta K_\alpha^\mu(y)} \frac{\delta}{\delta C_\mu^\alpha(y)} \right. \end{aligned} \quad (11.124)$$

$$\begin{aligned} &\quad \left. + \frac{\delta \bar{\Gamma}_0}{\delta L_\alpha(y)} \frac{\delta}{\delta \eta^\alpha(y)} + \frac{\delta \bar{\Gamma}_0}{\delta \eta^\alpha(y)} \frac{\delta}{\delta L_\alpha(y)} \right) \frac{\delta}{\delta \eta_{2\alpha\beta}(x)} \\ &\quad + \int d^4x d^4y \left[\frac{\delta L_{1\alpha}^\mu(x)}{\delta \eta^\sigma(y)} \frac{\delta \bar{\Gamma}_0}{\delta \eta_{1\alpha}^\mu(x)} \frac{\delta}{\delta L_\sigma(y)} + \frac{\delta L_{2\alpha\beta}(x)}{\delta \eta^\sigma(y)} \frac{\delta \bar{\Gamma}_0}{\delta \eta_{2\alpha\beta}(x)} \frac{\delta}{\delta L_\sigma(y)} \right], \\ \hat{g}^2 &= \int d^4x d^4y \left[\frac{\delta \bar{\Gamma}_0}{\delta C_\nu^\sigma(y)} \frac{\delta L_{1\alpha}^\mu(x)}{\delta K_\sigma^\nu(y)} \frac{\delta}{\delta \eta_{1\alpha}^\mu(x)} + \frac{\delta \bar{\Gamma}_0}{\delta L_\sigma(y)} \frac{\delta L_{1\alpha}^\mu(x)}{\delta \eta^\sigma(y)} \frac{\delta}{\delta \eta_{1\alpha}^\mu(x)} \right. \\ &\quad \left. + \frac{\delta \bar{\Gamma}_0}{\delta C_\mu^\sigma(y)} \frac{\delta L_{2\alpha\beta}(x)}{\delta K_\mu^\sigma(y)} \frac{\delta}{\delta \eta_{2\alpha\beta}(x)} + \frac{\delta \bar{\Gamma}_0}{\delta L_\sigma(y)} \frac{\delta L_{2\alpha\beta}(x)}{\delta \eta^\sigma(y)} \frac{\delta}{\delta \eta_{2\alpha\beta}(x)} \right] \\ &\quad + \int d^4x d^4y \left[\frac{\delta L_{1\alpha}^\mu(x)}{\delta \eta^\sigma(y)} \frac{\delta \bar{\Gamma}_0}{\delta \eta_{1\alpha}^\mu(x)} \frac{\delta}{\delta L_\sigma(y)} + \frac{\delta L_{2\alpha\beta}(x)}{\delta \eta^\sigma(y)} \frac{\delta \bar{\Gamma}_0}{\delta \eta_{2\alpha\beta}(x)} \frac{\delta}{\delta L_\sigma(y)} \right] \\ &\quad + \int d^4x d^4y \left[L_{1\alpha}^\mu(y) \left(\frac{\delta L_{1\sigma}^\nu(x)}{\delta \eta_{1\alpha}^\mu(y)} + L_{1\sigma}^\nu(x) \frac{\delta}{\delta \eta_{1\alpha}^\mu(y)} \right) \frac{\delta}{\delta \eta_{1\sigma}^\nu(x)} \right. \\ &\quad \left. + L_{1\alpha}^\mu(y) \left(\frac{\delta L_{2\rho\sigma}(x)}{\delta \eta_{1\alpha}^\mu(y)} + L_{2\rho\sigma}(x) \frac{\delta}{\delta \eta_{1\alpha}^\mu(y)} \right) \frac{\delta}{\delta \eta_{2\rho\sigma}(x)} \right. \\ &\quad \left. + L_{2\rho\sigma}(y) \left(\frac{\delta L_{1\alpha}^\mu(x)}{\delta \eta_{2\rho\sigma}(y)} + L_{1\alpha}^\mu(x) \frac{\delta}{\delta \eta_{2\rho\sigma}(y)} \right) \frac{\delta}{\delta \eta_{1\alpha}^\mu(x)} \right. \\ &\quad \left. + L_{2\rho\sigma}(y) \left(\frac{\delta L_{2\alpha\beta}(x)}{\delta \eta_{2\rho\sigma}(y)} + L_{2\alpha\beta}(x) \frac{\delta}{\delta \eta_{2\rho\sigma}(y)} \right) \frac{\delta}{\delta \eta_{2\alpha\beta}(x)} \right] \end{aligned} \quad (11.125)$$

Now, we try to determine the general solutions to eq.(11.112) and eq.(11.118). First, let's see the action $S[C, \eta_1, \eta_2]$ which is invariant under local gravitational gauge transformation. $S[C, \eta_1, \eta_2]$ is also invariant under generalized BRST transformation. The generalized BRST transformation of $S[C, \eta_1, \eta_2]$ is

$$\delta S[C, \eta_1, \eta_2] = \int d^4x \left(\delta C_\mu^\alpha(x) \frac{\delta S}{\delta C_\mu^\alpha(x)} + \delta \eta_{1\alpha}^\mu(x) \frac{\delta S}{\delta \eta_{1\alpha}^\mu(x)} + \delta \eta_{2\alpha\beta}(x) \frac{\delta S}{\delta \eta_{2\alpha\beta}(x)} \right). \quad (11.126)$$

Using generalized BRST transformation relation eqs.(11.1), (11.5-6), we can get the following relation,

$$\delta S[C, \eta_1, \eta_2] = \int d^4x \left(-\mathbf{D}_{\mu\beta}^\alpha \eta^\beta \frac{\delta S}{\delta C_\mu^\alpha(x)} - L_{1\alpha}^\mu(x) \frac{\delta S}{\delta \eta_{1\alpha}^\mu(x)} - L_{2\alpha\beta}(x) \frac{\delta S}{\delta \eta_{2\alpha\beta}(x)} \right) \delta \lambda. \quad (11.127)$$

Using eq.(11.48) and eq.(11.116), we can change the above relation into another form

$$\delta S[C, \eta_1, \eta_2] = \int d^4x \left(-\frac{\delta \bar{\Gamma}_0}{\delta K_\alpha^\mu} \frac{\delta S}{\delta C_\mu^\alpha(x)} - L_{1\alpha}^\mu(x) \frac{\delta S}{\delta \eta_{1\alpha}^\mu(x)} - L_{2\alpha\beta}(x) \frac{\delta S}{\delta \eta_{2\alpha\beta}(x)} \right) \delta \lambda. \quad (11.128)$$

Because $C[C, \eta_1, \eta_2]$ is a functional of only gravitational gauge fields C_μ^α , $\eta_{1\alpha}^\mu$ and $\eta_{2\alpha\beta}$, its functional derivatives to K_α^μ , L_α and η^α vanish

$$\frac{\delta S[C, \eta_1, \eta_2]}{\delta K_\alpha^\mu(x)} = 0, \quad (11.129)$$

$$\frac{\delta S[C, \eta_1, \eta_2]}{\delta L_\alpha(x)} = 0, \quad (11.130)$$

$$\frac{\delta S[C, \eta_1, \eta_2]}{\delta \eta^\alpha(x)} = 0. \quad (11.131)$$

Using these relations, we can prove that

$$\hat{g}S[C, \eta_1, \eta_2] = \int d^4x \left(\frac{\delta \bar{\Gamma}_0}{\delta K_\alpha^\mu} \frac{\delta S}{\delta C_\mu^\alpha(x)} + L_{1\alpha}^\mu(x) \frac{\delta S}{\delta \eta_{1\alpha}^\mu(x)} + L_{2\alpha\beta}(x) \frac{\delta S}{\delta \eta_{2\alpha\beta}(x)} \right) \quad (11.132)$$

Compare eq.(11.132) with eq.(11.128), we can get

$$\delta S[C, \eta_1, \eta_2] = -\hat{g}S[C, \eta_1, \eta_2] \delta \lambda. \quad (11.133)$$

The generalized BRST symmetry of $S[C, \eta_1, \eta_2]$ gives out the following important property of operator \hat{g} ,

$$\hat{g}S[C, \eta_1, \eta_2] = 0. \quad (11.134)$$

From eq.(11.134), we know that action $S[C, \eta_1, \eta_2]$ is a possible solution to eq.(11.122). Suppose that there is another functional f' which is functional of $C_\mu^\alpha(x)$, $\eta^\alpha(x)$, $\bar{\eta}_\alpha(x)$, and $K_\alpha^\mu(x)$,

$$f' = f'[C, \eta, \bar{\eta}, K]. \quad (11.135)$$

Because

$$\frac{\delta f'}{\delta \eta_{1\alpha}^\mu} = 0, \quad (11.136)$$

$$\frac{\delta f'}{\delta \eta_{2\alpha\beta}} = 0, \quad (11.137)$$

$$\frac{\delta f'}{\delta L_\sigma} = 0, \quad (11.138)$$

from eq.(11.124), we can prove that f' satisfies

$$\hat{g}_1 f'[C, \eta, \bar{\eta}, K] = 0. \quad (11.139)$$

So, $\hat{g}_1 f'$ is also a solution to eq.(11.122). The most general solution to eq.(11.122) can be written in the following form

$$\bar{\Gamma}_{N+1,div}^{[N]} = \alpha_{N+1}(\varepsilon)S[C, \eta_1, \eta_2] + \hat{g}_1 f'[C, \eta, \bar{\eta}, K] + f[C, \eta, \bar{\eta}, K, L, \eta_1, \eta_2], \quad (11.140)$$

where $f[C, \eta, \bar{\eta}, K, L, \eta_1, \eta_2]$ is an arbitrary functional of fields $C_\mu^\alpha(x)$, $\eta^\alpha(x)$, $\bar{\eta}_\alpha(x)$ and external sources $K_\alpha^\mu(x)$ and $L_\alpha(x)$, and $\eta_{1\alpha}^\mu(x)$ and $\eta_{2\alpha\beta}(x)$.

Now, let's consider constrains from eq.(11.119). Using eq.(11.129) and eq.(11.131), we can see that $S[C, \eta_1, \eta_2]$ satisfies eq.(11.119), so eq.(11.119) has no constrain on $S[C, \eta_1, \eta_2]$. Define a new variable

$$B_\alpha^\mu = K_\alpha^\mu - \partial^\mu \bar{\eta}_\alpha. \quad (11.141)$$

$f_1[B]$ is an arbitrary functional of B . It can be proved that

$$\frac{\delta f_1[B]}{\delta B_\alpha^\mu(x)} = \frac{\delta f_1[B]}{\delta K_\alpha^\mu(x)}, \quad (11.142)$$

$$\frac{\delta f_1[B]}{\delta \bar{\eta}_\alpha(x)} = \partial^\mu \frac{\delta f_1[B]}{\delta B_\alpha^\mu(x)}. \quad (11.143)$$

Combine these two relations, we will get

$$\frac{\delta f_1[B]}{\delta \bar{\eta}_\alpha(x)} = \partial^\mu \frac{\delta f_1[B]}{\delta K_\alpha^\mu(x)}. \quad (11.144)$$

There $f_1[B]$ is a solution to eq.(11.119). Suppose that there is another functional f_2 that is given by,

$$f_2[K, C, \eta] = \int d^4x K_\alpha^\mu T_\mu^\alpha(C, \eta), \quad (11.145)$$

where T_μ^α is a conserved current

$$\partial^\mu T_\mu^\alpha = 0. \quad (11.146)$$

It can be easily proved that $f_2[K, C, \eta]$ is also a solution of eq.(11.119). Because $\bar{\Gamma}_0$ satisfies eq.(11.119) (please see eq.(11.101)), operator \hat{g} commutes with $\frac{\delta}{\delta \bar{\eta}_\alpha(x)} - \partial^\mu \frac{\delta}{\delta K_\alpha^\mu(x)}$. It means that functional $f'[C, \eta, \bar{\eta}, K, L]$ in eq.(11.140) must satisfy eq.(11.119). According to these discussion, the solution of $f'[C, \eta, \bar{\eta}, K]$ has the following form,

$$f'[C, \eta, \bar{\eta}, K] = f_1[C, \eta, K_\alpha^\mu - \partial^\mu \bar{\eta}_\alpha] + \int d^4x K_\alpha^\mu T_\mu^\alpha(C, \eta). \quad (11.147)$$

In order to determine $f'[C, \eta, \bar{\eta}, K]$, we need to study dimensions of various fields and external sources. Set the dimensionality of mass to 1, i.e.

$$D[\hat{P}_\mu] = 1. \quad (11.148)$$

Then we have

$$D[C_\mu^\alpha] = 1, \quad (11.149)$$

$$D[d^4x] = -4, \quad (11.150)$$

$$D[D_\mu] = 1, \quad (11.151)$$

$$D[\eta] = D[\bar{\eta}] = 1, \quad (11.152)$$

$$D[K] = D[L] = 2, \quad (11.153)$$

$$D[g] = -1, \quad (11.154)$$

$$D[\eta_1] = D[\eta_2] = 0, \quad (11.155)$$

$$D[\bar{\Gamma}_{N+1,div}^{[N]}] = D[S] = 0. \quad (11.156)$$

Using these relations, we can prove that

$$D[\hat{g}] = 1, \quad (11.157)$$

$$D[f'] = -1. \quad (11.158)$$

Define virtual particle number N_g of ghost field η is 1, and that of ghost field $\bar{\eta}$ is -1, i.e.

$$N_g[\eta] = 1, \quad (11.159)$$

$$N_g[\bar{\eta}] = -1. \quad (11.160)$$

The virtual particle number is a additive conserved quantity, so Lagrangian and action carry no virtual particle number,

$$N_g[S] = N_g[\mathcal{L}] = 0. \quad (11.161)$$

The virtual particle number N_g of other fields and external sources are

$$N_g[C] = N_g[D_\mu] = 0, \quad (11.162)$$

$$N_g[g] = 0, \quad (11.163)$$

$$N_g[\bar{\Gamma}] = 0, \quad (11.164)$$

$$N_g[K] = -1, \quad (11.165)$$

$$N_g[L] = -2. \quad (11.166)$$

$$N_g[\eta_1] = N_g[\eta_2] = 0, \quad (11.167)$$

Using all these relations, we can determine the virtual particle number N_g of \hat{g} and f' ,

$$N_g[\hat{g}] = 1, \quad (11.168)$$

$$N_g[f'] = -1. \quad (11.169)$$

According to eq.(11.158) and eq.(11.169), we know that the dimensionality of f' is -1 and its virtual particle number if also -1 . Besides, f' must be a Lorentz scalar. Combine all these results, the only two possible solutions of $f_1[C, \eta, K_\alpha^\mu - \partial^\mu \bar{\eta}_\alpha]$ in eq.(11.124) are

$$(K_\alpha^\mu - \partial^\mu \bar{\eta}_\alpha) C_\mu^\alpha, \quad (11.170)$$

The only possible solution of T_μ^α is C_μ^α . But in general gauge conditions, C_μ^α does not satisfy the conserved equation eq.(11.146). Therefore, the solution to $f'[C, \eta, \bar{\eta}, K]$ is

$$f'[C, \eta, \bar{\eta}, K, L] = \int d^4x \left[\beta_{N+1}(\varepsilon) (K_\alpha^\mu - \partial^\mu \bar{\eta}_\alpha) C_\mu^\alpha \right], \quad (11.171)$$

where $\varepsilon = (4 - D)$, $\beta_{N+1}(\varepsilon)$ is divergent parameter when ε approaches zero. Then using the definition of \hat{g} , we can obtain the following result,

$$\hat{g}f'[C, \eta, \bar{\eta}, K, L] = -\beta_{N+1}(\varepsilon) \int d^4x \left[\frac{\delta \bar{\Gamma}_0}{\delta C_\mu^\alpha(x)} C_\mu^\alpha(x) + \frac{\delta \bar{\Gamma}_0}{\delta K_\alpha^\mu(x)} K_\alpha^\mu(x) - \bar{\eta}_\alpha \partial^\mu \mathbf{D}_{\mu\beta}^\alpha \eta^\beta \right]. \quad (11.172)$$

Therefore, the most general solution of $\bar{\Gamma}_{N+1,div}^{[N]}$ is

$$\begin{aligned} \bar{\Gamma}_{N+1,div}^{[N]} &= \alpha_{N+1}(\varepsilon) S[C, \eta_1, \eta_2] - \int d^4x \left[\beta_{N+1}(\varepsilon) \frac{\delta \bar{\Gamma}_0}{\delta C_\mu^\alpha(x)} C_\mu^\alpha(x) \right. \\ &\quad \left. + \beta_{N+1}(\varepsilon) \frac{\delta \bar{\Gamma}_0}{\delta K_\alpha^\mu(x)} K_\alpha^\mu(x) - \beta_{N+1}(\varepsilon) \bar{\eta}_\alpha \partial^\mu \mathbf{D}_{\mu\beta}^\alpha \eta^\beta \right]. \end{aligned} \quad (11.173)$$

In fact, the action $S[C, \eta_1, \eta_2]$ is a functional of pure gravitational gauge field. It also contains gravitational coupling constant g . So, we can denote it as $S[C, g]$,

$$S[C, g] = S[C, \eta_1, \eta_2]. \quad (11.174)$$

From eq.(4.20), eq.(4.24) and eq.(4.25), we can prove that the action $S[C, g]$ has the following important properties,

$$S[gC, 1] = g^2 S[C, g]. \quad (11.175)$$

Differentiate both sides of eq.(11.175) with respect to coupling constant g , we can get

$$S[C, g] = \frac{1}{2} \int d^4x \frac{\delta S[C, g]}{\delta C_\mu^\alpha(x)} C_\mu^\alpha(x) - \frac{1}{2} g \frac{\partial S[C, g]}{\partial g}. \quad (11.176)$$

It can be easily proved that

$$\int d^4x C_\mu^\alpha(x) \frac{\delta}{\delta C_\mu^\alpha(x)} \left[\int d^4y \bar{\eta}_\alpha(y) \partial^\mu \mathbf{D}_{\mu\beta}^\alpha \eta^\beta(y) \right] \quad (11.177)$$

$$= \int d^4x \left[(\partial^\mu \bar{\eta}_\beta(x)) (\partial_\mu \eta^\beta(x)) + \bar{\eta}_\alpha(x) \partial^\mu \mathbf{D}_{\mu\beta}^\alpha \eta^\beta(x) \right],$$

$$\int d^4x C_\mu^\alpha(x) \frac{\delta}{\delta C_\mu^\alpha(x)} \left[\int d^4y K_\alpha^\mu(y) \mathbf{D}_{\mu\beta}^\alpha \eta^\beta(y) \right] \quad (11.178)$$

$$= - \int d^4x \left[K_\alpha^\mu(x) \partial_\mu \eta^\alpha(x) - K_\alpha^\mu(x) \mathbf{D}_{\mu\beta}^\alpha \eta^\beta(x) \right],$$

$$g \frac{\partial}{\partial g} \left[\int d^4x \bar{\eta}_\alpha(x) \partial^\mu \mathbf{D}_{\mu\beta}^\alpha \eta^\beta(x) \right] \quad (11.179)$$

$$= \int d^4x \left[(\partial^\mu \bar{\eta}_\beta(x)) (\partial_\mu \eta^\beta(x)) + \bar{\eta}_\alpha(x) \partial^\mu \mathbf{D}_{\mu\beta}^\alpha \eta^\beta(x) \right],$$

$$g \frac{\partial}{\partial g} \left[\int d^4x K_\alpha^\mu(x) \mathbf{D}_{\mu\beta}^\alpha \eta^\beta(x) \right] \quad (11.180)$$

$$= - \int d^4x \left[K_\alpha^\mu(x) \partial_\mu \eta^\alpha - K_\alpha^\mu(x) \mathbf{D}_{\mu\beta}^\alpha \eta^\beta \right],$$

$$g \frac{\partial}{\partial g} \left[\int d^4x g L_\alpha(x) \eta^\beta(x) (\partial_\beta \eta^\alpha(x)) \right] \quad (11.181)$$

$$= \int d^4x g L_\alpha(x) \eta^\beta(x) (\partial_\beta \eta^\alpha(x)),$$

Using eqs.(11.177-178), eq.(11.116) and eq.(11.46), we can prove that

$$\begin{aligned} \int d^4x \frac{\delta S[C, g]}{\delta C_\mu^\alpha(x)} C_\mu^\alpha(x) &= \int d^4x \frac{\delta \bar{\Gamma}_0}{\delta C_\mu^\alpha(x)} C_\mu^\alpha(x) + \int d^4x [-(\partial^\mu \bar{\eta}_\alpha(x))(\partial_\mu \eta^\alpha(x)) \\ &\quad - \bar{\eta}_\alpha(x) \partial^\mu \mathbf{D}_{\mu\beta}^\alpha \eta^\beta(x) + K_\alpha^\mu(x) \partial_\mu \eta^\alpha(x) - K_\alpha^\mu(x) \mathbf{D}_{\mu\beta}^\alpha \eta^\beta(x)]. \end{aligned} \quad (11.182)$$

Similarly, we can get,

$$\begin{aligned} g \frac{\partial S[C, g]}{\partial g} &= g \frac{\delta \bar{\Gamma}_0}{\delta g} + \int d^4x \left[-(\partial^\mu \bar{\eta}_\alpha(x))(\partial_\mu \eta^\alpha(x)) - \bar{\eta}_\alpha(x) \partial^\mu \mathbf{D}_{\mu\beta}^\alpha \eta^\beta(x) \right. \\ &\quad \left. + K_\alpha^\mu(x) \partial_\mu \eta^\alpha(x) - K_\alpha^\mu(x) \mathbf{D}_{\mu\beta}^\alpha \eta^\beta(x) - g L_\alpha(x) \eta^\beta(x) (\partial_\beta \eta^\alpha(x)) \right]. \end{aligned} \quad (11.183)$$

Substitute eqs.(11.182-183) into eq.(11.176), we will get

$$S[C, g] = \frac{1}{2} \int d^4x \frac{\delta \bar{\Gamma}_0}{\delta C_\mu^\alpha(x)} C_\mu^\alpha(x) - \frac{1}{2} g \frac{\delta \bar{\Gamma}_0}{\delta g} + \int d^4x \left\{ \frac{1}{2} g L_\alpha(x) \eta^\beta(x) (\partial_\beta \eta^\alpha(x)) \right\}. \quad (11.184)$$

Substitute eq.(11.184) into eq.(11.173). we will get

$$\begin{aligned} \bar{\Gamma}_{N+1, div}^{[N]} &= \int d^4x \left[\left(\frac{\alpha_{N+1}(\varepsilon)}{2} - \beta_{N+1}(\varepsilon) \right) C_\mu^\alpha(x) \frac{\delta \bar{\Gamma}_0}{\delta C_\mu^\alpha(x)} \right. \\ &\quad \left. + \frac{\alpha_{N+1}(\varepsilon)}{2} L_\alpha(x) \frac{\delta \bar{\Gamma}_0}{\delta L_\alpha(x)} + \beta_{N+1}(\varepsilon) \bar{\eta}_\alpha(x) \frac{\delta \bar{\Gamma}_0}{\delta \bar{\eta}_\alpha(x)} \right. \\ &\quad \left. + \beta_{N+1}(\varepsilon) K_\alpha^\mu(x) \frac{\delta \bar{\Gamma}_0}{\delta K_\alpha^\mu(x)} \right] - \frac{\alpha_{N+1}(\varepsilon)}{2} g \frac{\delta \bar{\Gamma}_0}{\delta g} \end{aligned} \quad (11.185)$$

On the other hand, we can prove the following relations

$$\int d^4x \eta^\alpha(x) \frac{\delta}{\delta \eta^\alpha(x)} \left[\int d^4y \bar{\eta}_\beta(y) \partial^\mu \mathbf{D}_{\mu\sigma}^\beta \eta^\sigma(y) \right] = \int d^4x \bar{\eta}_\beta(x) \partial^\mu \mathbf{D}_{\mu\sigma}^\beta \eta^\sigma(x), \quad (11.186)$$

$$\int d^4x \eta^\alpha(x) \frac{\delta}{\delta \eta^\alpha(x)} \left[\int d^4y K_\beta^\mu(y) \mathbf{D}_{\mu\sigma}^\beta \eta^\sigma(y) \right] = \int d^4x K_\beta^\mu(x) \mathbf{D}_{\mu\sigma}^\beta \eta^\sigma(x), \quad (11.187)$$

$$\int d^4x \eta^\alpha(x) \frac{\delta}{\delta \eta^\alpha(x)} \left[\int d^4y g L_\beta(y) \eta^\sigma(y) (\partial_\sigma \eta^\beta(y)) \right] = 2 \int d^4x g L_\beta(x) \eta^\sigma(x) (\partial_\sigma \eta^\beta(x)), \quad (11.188)$$

$$\begin{aligned} \int d^4x \eta^\alpha(x) \frac{\delta \bar{\Gamma}_0}{\delta \eta^\alpha(x)} &= \int d^4x \left\{ \bar{\eta}_\beta(x) \partial^\mu \mathbf{D}_{\mu\sigma}^\beta \eta^\sigma(x) \right. \\ &\quad \left. + K_\beta^\mu(x) \mathbf{D}_{\mu\sigma}^\beta \eta^\sigma(x) + 2g L_\beta(x) \eta^\sigma(x) (\partial_\sigma \eta^\beta(x)) \right\}. \end{aligned} \quad (11.189)$$

Substitute eqs.(11.186-188) into eq.(11.189), we will get

$$\int d^4x \left\{ -\eta^\alpha \frac{\delta \bar{\Gamma}_0}{\delta \eta^\alpha} + \bar{\eta}_\alpha \frac{\delta \bar{\Gamma}_0}{\delta \bar{\eta}_\alpha} + K_\alpha^\mu \frac{\delta \bar{\Gamma}_0}{\delta K_\alpha^\mu} + 2L_\alpha \frac{\delta \bar{\Gamma}_0}{\delta L_\alpha} \right\} = 0. \quad (11.190)$$

Eq.(11.190) times $-\frac{\beta_{N+1}}{2}$, then add up this results and eq.(11.185), we will get

$$\begin{aligned}\bar{\Gamma}_{N+1,div}^{[N]} &= \int d^4x \left[\left(\frac{\alpha_{N+1}(\varepsilon)}{2} - \beta_{N+1}(\varepsilon) \right) \left(C_\mu^\alpha(x) \frac{\delta \bar{\Gamma}_0}{\delta C_\mu^\alpha(x)} + L_\alpha(x) \frac{\delta \bar{\Gamma}_0}{\delta L_\alpha(x)} \right) \right. \\ &\quad \left. + \frac{\beta_{N+1}(\varepsilon)}{2} \left(\eta^\alpha(x) \frac{\delta \bar{\Gamma}_0}{\delta \eta^\alpha(x)} + \bar{\eta}_\alpha(x) \frac{\delta \bar{\Gamma}_0}{\delta \bar{\eta}_\alpha(x)} + K_\alpha^\mu(x) \frac{\delta \bar{\Gamma}_0}{\delta K_\alpha^\mu(x)} \right) \right] \\ &\quad - \frac{\alpha_{N+1}(\varepsilon)}{2} g \frac{\delta \bar{\Gamma}_0}{\delta g}.\end{aligned}\quad (11.191)$$

This is the most general form of $\bar{\Gamma}_{N+1,div}^{[N]}$ which satisfies Ward-Takahashi identities.

According to minimal subtraction, the counterterm that cancel the divergent part of $\bar{\Gamma}_{N+1}^{[N]}$ is just $-\bar{\Gamma}_{N+1,div}^{[N]}$, that is

$$\tilde{S}^{[N+1]} = \tilde{S}^{[N]} - \bar{\Gamma}_{N+1,div}^{[N]} + o(\hbar^{N+2}), \quad (11.192)$$

where the term of $o(\hbar^{N+2})$ has no contribution to the divergences of $(N+1)$ -loop diagrams. Suppose that $\bar{\Gamma}_{N+1}^{[N+1]}$ is the generating functional of regular vertex which is calculated from $\tilde{S}^{[N+1]}$. It can be easily proved that

$$\bar{\Gamma}_{N+1}^{[N+1]} = \bar{\Gamma}_{N+1}^{[N]} - \bar{\Gamma}_{N+1,div}^{[N]}. \quad (11.193)$$

Using eq.(11.113), we can get

$$\bar{\Gamma}_{N+1}^{[N+1]} = \bar{\Gamma}_{N+1,F}^{[N]}. \quad (11.194)$$

$\bar{\Gamma}_{N+1}^{[N+1]}$ contains no divergence which is just what we expected.

Now, let's try to determine the form of $\tilde{S}^{[N+1]}$. Denote the non-renormalized action of the system as

$$\tilde{S}^{[0]} = \tilde{S}^{[0]} [C_\mu^\alpha, \bar{\eta}_\alpha, \eta^\alpha, K_\alpha^\mu, L_\alpha, g, \alpha, \eta_1, \eta_2]. \quad (11.195)$$

As one of the inductive hypothesis, we suppose that the action of the system after \hbar^N order renormalization is

$$\begin{aligned}\tilde{S}^{[N]} &= \tilde{S}^{[N]} [C_\mu^\alpha, \bar{\eta}_\alpha, \eta^\alpha, K_\alpha^\mu, L_\alpha, g, \alpha, \eta_1, \eta_2] \\ &= \tilde{S}^{[0]} \left[\sqrt{Z_1^{[N]}} C_\mu^\alpha, \sqrt{Z_2^{[N]}} \bar{\eta}_\alpha, \sqrt{Z_3^{[N]}} \eta^\alpha, \sqrt{Z_4^{[N]}} K_\alpha^\mu, \sqrt{Z_5^{[N]}} L_\alpha, Z_g^{[N]} g, Z_\alpha^{[N]} \alpha, Z_6^{[N]} \eta_1, Z_7^{[N]} \eta_2 \right].\end{aligned}\quad (11.196)$$

Substitute eq(11.191) and eq.(11.196) into eq.(11.192), we obtain

$$\begin{aligned}
\tilde{S}^{[N+1]} &= \tilde{S}^{[0]} \left[\sqrt{Z_1^{[N]}} C_\mu^\alpha, \sqrt{Z_2^{[N]}} \bar{\eta}_\alpha, \sqrt{Z_3^{[N]}} \eta^\alpha, \sqrt{Z_4^{[N]}} K_\alpha^\mu, \right. \\
&\quad \left. \sqrt{Z_5^{[N]}} L_\alpha, Z_g^{[N]} g, Z_\alpha^{[N]} \alpha, Z_6^{[N]} \eta_1, Z_7^{[N]} \eta_2 \right] \\
&- \int d^4x \left[\left(\frac{\alpha_{N+1}(\varepsilon)}{2} - \beta_{N+1}(\varepsilon) \right) \left(C_\mu^\alpha(x) \frac{\delta \bar{\Gamma}_0}{\delta C_\mu^\alpha(x)} + L_\alpha(x) \frac{\delta \bar{\Gamma}_0}{\delta L_\alpha(x)} \right) \right. \\
&\quad \left. + \frac{\beta_{N+1}(\varepsilon)}{2} \left(\eta^\alpha(x) \frac{\delta \bar{\Gamma}_0}{\delta \eta^\alpha(x)} + \bar{\eta}_\alpha(x) \frac{\delta \bar{\Gamma}_0}{\delta \bar{\eta}_\alpha(x)} + K_\alpha^\mu(x) \frac{\delta \bar{\Gamma}_0}{\delta K_\alpha^\mu(x)} \right) \right] \\
&+ \frac{\alpha_{N+1}(\varepsilon)}{2} g \frac{\partial \bar{\Gamma}_0}{\partial g} + o(\hbar^{N+2}).
\end{aligned} \tag{11.197}$$

Using eq.(11.116), we can prove that

$$\int d^4x C_\mu^\alpha(x) \frac{\delta \bar{\Gamma}_0}{\delta C_\mu^\alpha(x)} = \int d^4x C_\mu^\alpha(x) \frac{\delta \tilde{S}^{[0]}}{\delta C_\mu^\alpha(x)} + 2\alpha \frac{\partial \tilde{S}^{[0]}}{\partial \alpha}, \tag{11.198}$$

$$\int d^4x L_\alpha(x) \frac{\delta \bar{\Gamma}_0}{\delta L_\alpha(x)} = \int d^4x L_\alpha(x) \frac{\delta \tilde{S}^{[0]}}{\delta L_\alpha(x)}, \tag{11.199}$$

$$\int d^4x \bar{\eta}_\alpha(x) \frac{\delta \bar{\Gamma}_0}{\delta \bar{\eta}_\alpha(x)} = \int d^4x \bar{\eta}_\alpha(x) \frac{\delta \tilde{S}^{[0]}}{\delta \bar{\eta}_\alpha(x)}, \tag{11.200}$$

$$\int d^4x \eta^\alpha(x) \frac{\delta \bar{\Gamma}_0}{\delta \eta^\alpha(x)} = \int d^4x \eta^\alpha(x) \frac{\delta \tilde{S}^{[0]}}{\delta \eta^\alpha(x)}, \tag{11.201}$$

$$\int d^4x K_\alpha^\mu(x) \frac{\delta \bar{\Gamma}_0}{\delta K_\alpha^\mu(x)} = \int d^4x K_\alpha^\mu(x) \frac{\delta \tilde{S}^{[0]}}{\delta K_\alpha^\mu(x)}, \tag{11.202}$$

$$g \frac{\partial \bar{\Gamma}_0}{\partial g} = g \frac{\partial \tilde{S}^{[0]}}{\partial g}. \tag{11.203}$$

Using these relations, eq.(11.197) is changed into,

$$\begin{aligned}
\tilde{S}^{[N+1]} &= \tilde{S}^{[0]} [\sqrt{Z_1^{[N]}} C_\mu^\alpha, \sqrt{Z_2^{[N]}} \bar{\eta}_\alpha, \sqrt{Z_3^{[N]}} \eta^\alpha, \sqrt{Z_4^{[N]}} K_\alpha^\mu, \\
&\quad \sqrt{Z_5^{[N]}} L_\alpha, Z_g^{[N]} g, Z_\alpha^{[N]} \alpha, Z_6^{[N]} \eta_1, Z_7^{[N]} \eta_2] \\
&\quad - \int d^4x \left[\left(\frac{\alpha_{N+1}(\varepsilon)}{2} - \beta_{N+1}(\varepsilon) \right) \left(C_\mu^\alpha(x) \frac{\delta \tilde{S}^{[0]}}{\delta C_\mu^\alpha(x)} + L_\alpha(x) \frac{\delta \tilde{S}^{[0]}}{\delta L_\alpha(x)} \right) \right. \\
&\quad \left. + \frac{\beta_{N+1}(\varepsilon)}{2} \left(\eta^\alpha(x) \frac{\delta \tilde{S}^{[0]}}{\delta \eta^\alpha} + \bar{\eta}_\alpha(x) \frac{\delta \tilde{S}^{[0]}}{\delta \bar{\eta}_\alpha(x)} + K_\alpha^\mu(x) \frac{\delta \tilde{S}^{[0]}}{\delta K_\alpha^\mu(x)} \right) \right] \\
&\quad + \frac{\alpha_{N+1}(\varepsilon)}{2} g \frac{\partial \tilde{S}^{[0]}}{\partial g} - 2 \left(\frac{\alpha_{N+1}(\varepsilon)}{2} - \beta_{N+1}(\varepsilon) \right) \alpha \frac{\partial \tilde{S}^{[0]}}{\partial \alpha} + o(\hbar^{N+2})
\end{aligned} \tag{11.204}$$

We can see that this relation has just the form of first order functional expansion. Using this relation, we can determine the form of $\tilde{S}^{[N+1]}$. It is

$$\begin{aligned}
\tilde{S}^{[N+1]} &[C_\mu^\alpha, \bar{\eta}_\alpha, \eta^\alpha, K_\alpha^\mu, L_\alpha, g, \alpha, \eta_1, \eta_2] \\
&= \tilde{S}^{[0]} [\sqrt{Z_1^{[N+1]}} C_\mu^\alpha, \sqrt{Z_2^{[N+1]}} \bar{\eta}_\alpha, \sqrt{Z_3^{[N+1]}} \eta^\alpha, \sqrt{Z_4^{[N+1]}} K_\alpha^\mu, \\
&\quad \sqrt{Z_5^{[N+1]}} L_\alpha, Z_g^{[N+1]} g, Z_\alpha^{[N+1]} \alpha, Z_6^{[N+1]} \eta_1, Z_7^{[N+1]} \eta_2],
\end{aligned} \tag{11.205}$$

where

$$\sqrt{Z_1^{[N+1]}} = \sqrt{Z_5^{[N+1]}} = \sqrt{Z_\alpha^{[N+1]}} = 1 - \sum_{L=1}^{N+1} \left(\frac{\alpha_L(\varepsilon)}{2} - \beta_L(\varepsilon) \right), \tag{11.206}$$

$$\sqrt{Z_2^{[N+1]}} = \sqrt{Z_3^{[N+1]}} = \sqrt{Z_4^{[N+1]}} = 1 - \sum_{L=1}^{N+1} \frac{\beta_L(\varepsilon)}{2}, \tag{11.207}$$

$$Z_g^{[N+1]} = 1 + \sum_{L=1}^{N+1} \frac{\alpha_L(\varepsilon)}{2}, \tag{11.208}$$

$$Z_6^{[N+1]} = 1, \tag{11.209}$$

$$Z_7^{[N+1]} = 1. \tag{11.210}$$

Denote

$$\sqrt{Z^{[N+1]}} \triangleq \sqrt{Z_1^{[N+1]}} = \sqrt{Z_5^{[N+1]}} = \sqrt{Z_\alpha^{[N+1]}}, \tag{11.211}$$

$$\sqrt{\tilde{Z}^{[N+1]}} \triangleq \sqrt{Z_2^{[N+1]}} = \sqrt{Z_3^{[N+1]}} = \sqrt{Z_4^{[N+1]}}. \quad (11.212)$$

The eq.(11.205) is changed into

$$\begin{aligned} & \tilde{S}^{[N+1]} [C_\mu^\alpha, \bar{\eta}_\alpha, \eta^\alpha, K_\alpha^\mu, L_\alpha, g, \alpha, \eta_1, \eta_2] \\ &= \tilde{S}^{[0]} [\sqrt{Z^{[N+1]}} C_\mu^\alpha, \sqrt{\tilde{Z}^{[N+1]}} \bar{\eta}_\alpha, \sqrt{\tilde{Z}^{[N+1]}} \eta^\alpha, \sqrt{\tilde{Z}^{[N+1]}} K_\alpha^\mu, \\ & \quad \sqrt{Z^{[N+1]}} L_\alpha, Z_g^{[N+1]} g, Z^{[N+1]} \alpha, \eta_1, \eta_2]. \end{aligned} \quad (11.213)$$

Using eq.(11.213), we can easily prove that

$$\begin{aligned} & \bar{\Gamma}^{[N+1]} [C_\mu^\alpha, \bar{\eta}_\alpha, \eta^\alpha, K_\alpha^\mu, L_\alpha, g, \alpha, \eta_1, \eta_2] \\ &= \bar{\Gamma}^{[0]} \left[\sqrt{Z^{[N+1]}} C_\mu^\alpha, \sqrt{\tilde{Z}^{[N+1]}} \bar{\eta}_\alpha, \sqrt{\tilde{Z}^{[N+1]}} \eta^\alpha, \right. \\ & \quad \left. \sqrt{\tilde{Z}^{[N+1]}} K_\alpha^\mu, \sqrt{Z^{[N+1]}} L_\alpha, Z_g^{[N+1]} g, Z^{[N+1]} \alpha, \eta_1, \eta_2 \right]. \end{aligned} \quad (11.214)$$

$\eta^{\mu\nu}$ is the Minkowski metric of flat space-time. The functions of $\eta_{1\alpha}^\mu$ and $\eta_{2\alpha\beta}$ are also similar to those of metric, which is used to contract Lorentz indexes and group indexes. From eq.(11.214), we can see the renormalization of the theory does not affect their value. It means that normalization of the theory does not affect the space-time structure, which is consistent with our basic viewpoint that physical space-time in the gravitational gauge theory is always flat.

Now, we need to prove that all inductive hypotheses hold at $L = N + 1$. The main inductive hypotheses which is used in the above proof are the following three: when $L = N$, the following three hypotheses hold,

1. the lowest divergence of $\bar{\Gamma}^{[N]}$ appears in the $(N + 1)$ -loop diagram;
2. $\bar{\Gamma}^{[N]}$ satisfies Ward-Takahashi identities eqs.(11.106-107);
3. after \hbar^N th order renormalization, the action of the system has the form of eq.(11.196).

First, let's see the first hypothesis. According to eq.(11.194), after introducing $(N + 1)$ th order counterterm, the $(N + 1)$ -loop diagram contribution of $\bar{\Gamma}^{[N+1]}$ is finite. It means that the lowest order divergence of $\bar{\Gamma}^{[N+1]}$ appears in the $(N + 2)$ -loop diagram. So, the first inductive hypothesis hold when $L = N + 1$.

It can be proved that the renormalization constants $Z_g^{[N]}$, $Z^{[N]}$ and $\tilde{Z}^{[N]}$ satisfy the following relation,

$$Z_g^{[N]} \tilde{Z}^{[N]} \sqrt{Z^{[N]}} = 1, \quad (11.215)$$

where N is an arbitrary non-negative number. It is known that the non-renormalized generating functional of regular vertex

$$\bar{\Gamma}^{[0]} = \bar{\Gamma}^{[0]}[C, \bar{\eta}, \eta, K, L, g, \alpha, \eta_1, \eta_2] \quad (11.216)$$

satisfies Ward-Takahashi identities eqs.(11.101-102). If we define

$$\bar{\Gamma}' = \bar{\Gamma}^{[0]}[C', \bar{\eta}', \eta', K', L', g', \alpha', \eta_1, \eta_2], \quad (11.217)$$

then, it must satisfy the following Ward-Takahashi identities

$$\partial^\mu \frac{\delta \bar{\Gamma}'}{\delta K'_\alpha{}^\mu(x)} = \frac{\delta \bar{\Gamma}'}{\delta \bar{\eta}'_\alpha(x)}, \quad (11.218)$$

$$\int d^4x \left\{ \frac{\delta \bar{\Gamma}'}{\delta K'_\alpha{}^\mu(x)} \frac{\delta \bar{\Gamma}'}{\delta C'_\mu{}^\alpha(x)} + \frac{\delta \bar{\Gamma}'}{\delta L'_\alpha(x)} \frac{\delta \bar{\Gamma}'}{\delta \eta'^\alpha(x)} + L'_{1\alpha}{}^\mu(x) \frac{\delta \bar{\Gamma}'}{\delta \eta'_{1\alpha}{}^\mu(x)} + L'_{2\alpha\beta}(x) \frac{\delta \bar{\Gamma}'}{\delta \eta'_{2\alpha\beta}(x)} \right\} = 0, \quad (11.219)$$

where

$$L'_{1\alpha}{}^\mu = g' \eta'_{1\sigma}{}^\mu (\partial_\alpha \eta'^\sigma), \quad (11.220)$$

$$L'_{2\alpha\beta} = g' [\eta_{2\alpha\sigma} (\partial_\beta \eta'^\sigma) + \eta_{2\sigma\beta} (\partial_\alpha \eta'^\sigma)]. \quad (11.221)$$

Set,

$$C'_\mu{}^\alpha = \sqrt{Z^{[N+1]}} C_\mu^\alpha, \quad (11.222)$$

$$K'_\alpha{}^\mu = \sqrt{\tilde{Z}^{[N+1]}} K_\alpha^\mu, \quad (11.223)$$

$$L'_\alpha = \sqrt{Z^{[N+1]}} L_\alpha, \quad (11.224)$$

$$\eta'^\alpha = \sqrt{\tilde{Z}^{[N+1]}} \eta^\alpha, \quad (11.225)$$

$$\bar{\eta}'_\alpha = \sqrt{\tilde{Z}^{[N+1]}} \bar{\eta}_\alpha, \quad (11.226)$$

$$g' = Z_g^{[N+1]} g, \quad (11.227)$$

$$\alpha' = Z^{[N+1]} \alpha. \quad (11.228)$$

In this case, we have

$$L'_{1\alpha}{}^\mu = \sqrt{\tilde{Z}^{[N+1]}} Z_g^{[N+1]} L_{1\alpha}{}^\mu, \quad (11.229)$$

$$L'_{2\alpha\beta} = \sqrt{\tilde{Z}^{[N+1]}} Z_g^{[N+1]} L_{2\alpha\beta}, \quad (11.230)$$

$$\bar{\Gamma}' = \bar{\Gamma}^{[0]} \left[\sqrt{\tilde{Z}^{[N+1]}} C_\mu^\alpha, \sqrt{\tilde{Z}^{[N+1]}} \bar{\eta}_\alpha, \sqrt{\tilde{Z}^{[N+1]}} \eta^\alpha, \right. \\ \left. \sqrt{\tilde{Z}^{[N+1]}} K_\alpha^\mu, \sqrt{\tilde{Z}^{[N+1]}} L_\alpha, Z_g^{[N+1]} g, Z^{[N+1]} \alpha, , \eta_1, \eta_2 \right] \quad (11.231)$$

$$= \bar{\Gamma}^{[N+1]}.$$

Then eq.(11.218) is changed into

$$\frac{1}{\sqrt{\tilde{Z}^{[N+1]}}} \partial^\mu \frac{\delta \bar{\Gamma}^{[N+1]}}{\delta K_\alpha^\mu(x)} = \frac{1}{\sqrt{\tilde{Z}^{[N+1]}}} \frac{\delta \bar{\Gamma}^{[N+1]}}{\delta \bar{\eta}_\alpha(x)}. \quad (11.232)$$

Because $\frac{1}{\sqrt{\tilde{Z}^{[N+1]}}}$ does not vanish, the above equation gives out

$$\partial^\mu \frac{\delta \bar{\Gamma}^{[N+1]}}{\delta K_\alpha^\mu(x)} = \frac{\delta \bar{\Gamma}^{[N+1]}}{\delta \bar{\eta}_\alpha(x)}. \quad (11.233)$$

Eq.(11.219) gives out

$$\int d^4x \left\{ \frac{1}{\sqrt{\tilde{Z}^{[N+1]}} \sqrt{Z^{[N+1]}}} \left[\frac{\delta \bar{\Gamma}^{[N+1]}}{\delta K_\alpha^\mu(x)} \frac{\delta \bar{\Gamma}^{[N+1]}}{\delta C_\mu^\alpha(x)} + \frac{\delta \bar{\Gamma}^{[N+1]}}{\delta L_\alpha(x)} \frac{\delta \bar{\Gamma}^{[N+1]}}{\delta \eta^\alpha(x)} \right] \right. \\ \left. + \sqrt{\tilde{Z}^{[N+1]}} Z_g^{[N+1]} \left[L_{1\alpha}^\mu(x) \frac{\delta \bar{\Gamma}^{[N+1]}}{\delta \eta_{1\alpha}^\mu(x)} + L_{2\alpha\beta}(x) \frac{\delta \bar{\Gamma}^{[N+1]}}{\delta \eta_{2\alpha\beta}(x)} \right] \right\} = 0. \quad (11.234)$$

Using eq.(11.215), we will get

$$\int d^4x \left\{ \frac{\delta \bar{\Gamma}^{[N+1]}}{\delta K_\alpha^\mu(x)} \frac{\delta \bar{\Gamma}^{[N+1]}}{\delta C_\mu^\alpha(x)} + \frac{\delta \bar{\Gamma}^{[N+1]}}{\delta L_\alpha(x)} \frac{\delta \bar{\Gamma}^{[N+1]}}{\delta \eta^\alpha(x)} + L_{1\alpha}^\mu(x) \frac{\delta \bar{\Gamma}^{[N+1]}}{\delta \eta_{1\alpha}^\mu(x)} + L_{2\alpha\beta}(x) \frac{\delta \bar{\Gamma}^{[N+1]}}{\delta \eta_{2\alpha\beta}(x)} \right\} = 0. \quad (11.235)$$

Eq.(11.233) and eq.(11.235) are just the Ward-Takahashi identities for $L = N + 1$. Therefore, the second inductive hypothesis holds when $L = N + 1$.

The third inductive hypothesis has already been proved which is shown in eq.(11.213). Therefore, all three inductive hypothesis hold when $L = N + 1$. According to inductive principle, they will hold when L is an arbitrary non-negative number, especially they hold when L approaches infinity.

In above discussions, we have proved that, if we suppose that when $L = N$ eq.(11.196) holds, then it also holds when $L = N + 1$. According to inductive principle, we know that eq.(11.213-214) hold for any positive integer N . When N approaches infinity, we get

$$\begin{aligned} & \tilde{S} [C_\mu^\alpha, \bar{\eta}_\alpha, \eta^\alpha, K_\alpha^\mu, L_\alpha, g, \alpha, \eta_1, \eta_2] \\ &= \tilde{S}^{[0]} \left[\sqrt{\tilde{Z}} C_\mu^\alpha, \sqrt{\tilde{Z}} \bar{\eta}_\alpha, \sqrt{\tilde{Z}} \eta^\alpha, \sqrt{\tilde{Z}} K_\alpha^\mu, \sqrt{\tilde{Z}} L_\alpha, Z_g g, Z\alpha, \eta_1, \eta_2 \right], \end{aligned} \quad (11.236)$$

$$\begin{aligned} & \bar{\Gamma} [C_\mu^\alpha, \bar{\eta}_\alpha, \eta^\alpha, K_\alpha^\mu, L_\alpha, g, \alpha, \eta_1, \eta_2] \\ &= \bar{\Gamma}^{[0]} \left[\sqrt{\tilde{Z}} C_\mu^\alpha, \sqrt{\tilde{Z}} \bar{\eta}_\alpha, \sqrt{\tilde{Z}} \eta^\alpha, \sqrt{\tilde{Z}} K_\alpha^\mu, \sqrt{\tilde{Z}} L_\alpha, Z_g g, Z\alpha, \eta_1, \eta_2 \right], \end{aligned} \quad (11.237)$$

where

$$\sqrt{\tilde{Z}} = \lim_{N \rightarrow \infty} \sqrt{\tilde{Z}^{[N]}} = 1 - \sum_{L=1}^{\infty} \left(\frac{\alpha_L(\varepsilon)}{2} - \beta_L(\varepsilon) \right), \quad (11.238)$$

$$\sqrt{\tilde{Z}} = \lim_{N \rightarrow \infty} \sqrt{\tilde{Z}^{[N]}} = 1 - \sum_{L=1}^{\infty} \frac{\beta_L(\varepsilon)}{2}, \quad (11.239)$$

$$Z_g = \lim_{N \rightarrow \infty} Z_g^{[N]} = 1 + \sum_{L=1}^{\infty} \frac{\alpha_L(\varepsilon)}{2}. \quad (11.240)$$

$\tilde{S} [C_\mu^\alpha, \bar{\eta}_\alpha, \eta^\alpha, K_\alpha^\mu, L_\alpha, g, \alpha, \eta_1, \eta_2]$ and $\bar{\Gamma} [C_\mu^\alpha, \bar{\eta}_\alpha, \eta^\alpha, K_\alpha^\mu, L_\alpha, g, \alpha, \eta_1, \eta_2]$ are renormalized action and generating functional of regular vertex. The generating functional of regular vertex $\bar{\Gamma}$ contains no divergence. All kinds of vertex that generated from $\bar{\Gamma}$ are finite. From eq.(11.236) and eq.(11.237), we can see that we only introduce three unknown parameters which are $\sqrt{\tilde{Z}}$, $\sqrt{\tilde{Z}}$ and Z_g . Therefore, gravitational gauge theory is a renormalizable theory.

We have already proved that the renormalized generating functional of regular vertex satisfies Ward-Takahashi identities,

$$\partial^\mu \frac{\delta \bar{\Gamma}}{\delta K_\alpha^\mu(x)} = \frac{\delta \bar{\Gamma}}{\delta \bar{\eta}_\alpha(x)}, \quad (11.241)$$

$$\int d^4x \left\{ \frac{\delta \bar{\Gamma}}{\delta K_\alpha^\mu(x)} \frac{\delta \bar{\Gamma}}{\delta C_\mu^\alpha(x)} + \frac{\delta \bar{\Gamma}}{\delta L_\alpha(x)} \frac{\delta \bar{\Gamma}}{\delta \eta^\alpha(x)} + L_{1\alpha}^\mu \frac{\delta \bar{\Gamma}}{\delta \eta_{1\alpha}^\mu(x)} + L_{2\alpha\beta} \frac{\delta \bar{\Gamma}}{\delta \eta_{2\alpha\beta}(x)} \right\} = 0. \quad (11.242)$$

It means that the renormalized theory has the structure of gauge symmetry. If we define

$$C_{0\mu}^\alpha = \sqrt{\tilde{Z}} C_\mu^\alpha, \quad (11.243)$$

$$\eta_0^\alpha = \sqrt{\tilde{Z}} \eta^\alpha, \quad (11.244)$$

$$\bar{\eta}_{0\alpha} = \sqrt{\tilde{Z}} \bar{\eta}_\alpha, \quad (11.245)$$

$$K_{0\alpha}^\mu = \sqrt{\tilde{Z}} K_\alpha^\mu, \quad (11.246)$$

$$L_{0\alpha} = \sqrt{\tilde{Z}} L_\alpha, \quad (11.247)$$

$$g_0 = Z_g g, \quad (11.248)$$

$$\alpha_0 = Z \alpha. \quad (11.249)$$

Therefore, eqs.(11.236-237) are changed into

$$\tilde{S} [C_\mu^\alpha, \bar{\eta}_\alpha, \eta^\alpha, K_\alpha^\mu, L_\alpha, g, \alpha, \eta_1, \eta_2] = \tilde{S}^{[0]} [C_{0\mu}^\alpha, \bar{\eta}_{0\alpha}, \eta_0^\alpha, K_{0\alpha}^\mu, L_{0\alpha}, g_0, \alpha_0, \eta_1, \eta_2], \quad (11.250)$$

$$\bar{\Gamma}[C_\mu^\alpha, \bar{\eta}_\alpha, \eta^\alpha, K_\alpha^\mu, L_\alpha, g, \alpha, \eta_1, \eta_2] = \bar{\Gamma}^{[0]}[C_{0\mu}^\alpha, \bar{\eta}_{0\alpha}, \eta_0^\alpha, K_{0\alpha}^\mu, L_{0\alpha}, g_0, \alpha_0, \eta_1, \eta_2]. \quad (11.251)$$

$C_{0\mu}^\alpha$, $\bar{\eta}_{0\alpha}$ and η_0^α are renormalized wave function, $K_{0\alpha}^\mu$ and $L_{0\alpha}$ are renormalized external sources, and g_0 is the renormalized gravitational coupling constant.

The action \tilde{S} which is given by eq.(11.250) is invariant under the following generalized BRST transformations,

$$\delta C_{0\mu}^\alpha = -\mathbf{D}_{0\mu\ \beta}^\alpha \eta_0^\beta \delta\lambda, \quad (11.252)$$

$$\delta \eta_0^\alpha = g_0 \eta_0^\sigma (\partial_\sigma \eta_0^\alpha) \delta\lambda, \quad (11.253)$$

$$\delta \bar{\eta}_{0\alpha} = \frac{1}{\alpha_0} \eta_{\alpha\beta} f_0^\beta \delta\lambda, \quad (11.254)$$

$$\delta \eta^{\mu\nu} = 0, \quad (11.255)$$

$$\delta \eta_{1\alpha}^\mu = -g_0 \eta_{1\sigma}^\mu (\partial_\alpha \eta_0^\sigma) \delta\lambda, \quad (11.256)$$

$$\delta \eta_{2\alpha\beta} = -g_0 (\eta_{2\alpha\sigma} (\partial_\beta \eta_0^\sigma) + \eta_{2\sigma\beta} (\partial_\alpha \eta_0^\sigma)) \delta\lambda, \quad (11.257)$$

where,

$$\mathbf{D}_{0\mu\beta}^\alpha = \delta_\beta^\alpha \partial_\mu - g_0 \delta_\beta^\alpha C_{0\mu}^\sigma \partial_\sigma + g_0 (\partial_\beta C_{0\mu}^\alpha), \quad (11.258)$$

$$f_0^\alpha = \partial^\mu C_{0\mu}^\alpha. \quad (11.259)$$

Therefore, the normalized action has generalized BRST symmetry, which means that the normalized theory has the structure of gauge theory.

12 Theoretical Predictions

The gravitational gauge field theory which is discussed in this paper is renormalizable. Its transcendental foundation is gauge principle. Gravitational gauge interactions is completely determined by gauge symmetry. In other words, the Lagrangian of the system is completely determined by gauge symmetry. Anyone who is familiar with traditional quantum gravity must have realized that gravitational gauge theory is quite different from the traditional quantum gravity. In this chapter, we plan to discuss some predictions of the theory which is useful for experimental research and is useful for testing the validity of the theory. Now, let's discuss them one by one.

1. [**Gravitational Wave**] In gravitational gauge theory, the gravitational gauge field is represented by C_μ . From the point of view of quantum field theory, gravitational gauge field C_μ is a vector field and it obeys dynamics of vector field. In other words, gravitational wave is vector wave. Suppose that the gravitational gauge field is very weak in vacuum, then in leading order approximation, the equation of motion of gravitational wave is

$$\partial^\mu F_{0\mu\nu}^\alpha = 0, \quad (12.1)$$

where $F_{0\mu\nu}^\alpha$ is given by eq.(5.9). If we set gC_μ^α equals zero, we can obtain eq.(12.1) from eq.(4.39). Eq.(12.1) is very similar to the famous Maxwell equation in vacuum. Define

$$F_{ij}^\alpha = -\varepsilon_{ijk} B_k^\alpha, \quad F_{0i}^\alpha = E_i^\alpha, \quad (12.2)$$

then eq.(12.1) is changed into

$$\nabla \cdot \vec{E}^\alpha = 0, \quad (12.3)$$

$$\frac{\partial}{\partial t} \vec{E}^\alpha - \nabla \times \vec{B}^\alpha = 0. \quad (12.4)$$

From definitions eq.(12.2), we can prove that

$$\nabla \cdot \vec{B}^\alpha = 0, \quad (12.5)$$

$$\frac{\partial}{\partial t} \vec{B}^\alpha + \nabla \times \vec{E}^\alpha = 0. \quad (12.6)$$

If there were no superscript α , eqs.(12.3-6) would be the ordinary Maxwell equations. In ordinary case, the strength of gravitational field in vacuum is extremely weak, so the gravitational wave in vacuum is composed of four independent vector waves.

Though gravitational gauge field is a vector field, its component fields C_μ^α have one Lorentz index μ and one group index α . Both indexes have the same behavior under Lorentz transformation. According to the behavior of Lorentz transformation, gravitational field likes a tensor field. We call it pseudo-tensor field. The spin of a field is determined according to its behavior under Lorentz transformation, so the spin of gravitational field is 2. In conventional quantum field theory, spin-1 field is a vector field, and vector field is a spin-1 field. In gravitational gauge field, this correspondence is violated. The reason is that, in gravitational gauge field theory, the group index contributes to the spin of a field, while in ordinary gauge field theory, the group index do not contribute to the spin of a field. In a word, gravitational field is a spin-2 vector field.

2. [**Gravitational Magnetic Field**] From eq.(12.3-6), we can see that the equations of motion of gravitational wave in vacuum are quite similar to those of electromagnetic wave. The phenomenological behavior of gravitational wave must also be similar to that of electromagnetic wave. In gravitational gauge theory, \vec{B}^α is called the gravitational magnetic field. It will transmit gravitational magnetic interactions between two rotating objects. In first order approximation, the equation of motion of gravitational gauge field is

$$\partial^\mu F_{\mu\nu}^\alpha = -g\eta_{\nu\tau}\eta_2^{\alpha\beta}T_{g\beta}^\tau. \quad (12.7)$$

Using eq.(12.2) and eq.(12.7), we can get the following equations

$$\nabla \cdot \vec{E}^\alpha = -g\eta_2^{\alpha\beta}T_{g\beta}^0, \quad (12.8)$$

$$\frac{\partial}{\partial t} \vec{E}^\alpha - \nabla \times \vec{B}^\alpha = +g\eta_2^{\alpha\beta} \vec{T}_{g\beta}, \quad (12.9)$$

where $\vec{T}_{g\beta}$ is a simplified notation whose explicit definition is given by the following relation

$$(\vec{T}_{g\beta})^i = \vec{T}_{g\beta}^i. \quad (12.10)$$

On the other hand, it is easy to prove that(omit self interactions of graviton)

$$\partial_\mu F_{\nu\lambda}^\alpha + \partial_\nu F_{\lambda\mu}^\alpha + \partial_\lambda F_{\mu\nu}^\alpha = 0. \quad (12.11)$$

From eq.(12.11), we can get

$$\nabla \cdot \vec{B}^\alpha = 0, \quad (12.12)$$

$$\frac{\partial}{\partial t} \vec{B}^\alpha + \nabla \times \vec{E}^\alpha = 0. \quad (12.13)$$

Eq.(12.8) means that energy-momentum density of the system is the source of gravitational electric fields, eq.(12.9) means that time-varying gravitational electric fields give rise to gravitational magnetic fields, and eq.(12.13) means that time-varying gravitational magnetic fields give rise to gravitational electric fields. Suppose that the angular momentum of an rotating object is J_i , then there will be a coupling between angular momentum and gravitational magnetic fields. The interaction Hamiltonian of this coupling is proportional to $(\frac{P_\alpha}{m} \vec{J} \cdot \vec{B}^\alpha)$. The existence of gravitational magnetic fields is important for cosmology. It is known that almost all galaxies in the universe rotate. The global rotation of galaxy will give rise to gravitational magnetic fields in space-time. The existence of gravitational magnetic fields will affect the moving of stars in (or near) the galaxy. This influence contributes to the formation of the galaxy and can explain why almost all galaxies have global large scale structures. In other words, the gravitational magnetic fields contribute great to the large scale structure of galaxy and universe.

3. [**Lorentz Force**] There is a force when a particle is moving in a gravitational magnetic field. In electromagnetic field theory, this force is usually called Lorentz force. As an example, we discuss gravitational interactions between gravitational field and Dirac field. Suppose that the gravitational field is static. According to eqs.(6.2-3), the interaction Lagrangian is

$$\mathcal{L}_I = ge^{I(C)} \bar{\psi} \gamma^\mu \partial_\alpha \psi C_\mu^\alpha. \quad (12.14)$$

For Dirac field, the gravitational energy-momentum of Dirac field is

$$T_{g\alpha}^\mu = \bar{\psi} \gamma^\mu \partial_\alpha \psi. \quad (12.15)$$

Substitute eq.(12.15) into eq.(12.14), we get

$$\mathcal{L}_I = ge^{I(C)} T_{g\alpha}^\mu C_\mu^\alpha. \quad (12.16)$$

The interaction Hamiltonian density \mathcal{H}_I is

$$\mathcal{H}_I = -\mathcal{L}_I = -ge^{I(C)} T_{g\alpha}^\mu(y, \vec{x}) C_\mu^\alpha(y). \quad (12.17)$$

Suppose that the moving particle is a mass point at point \vec{x} , in this case

$$T_{g\alpha}^\mu(y, \vec{x}) = T_{g\alpha}^\mu \delta(\vec{y} - \vec{x}), \quad (12.18)$$

where $T_{g\alpha}^\mu$ is independent of space coordinates. Then, the interaction Hamiltonian H_I is

$$H_I = \int d^3 \vec{y} \mathcal{H}_I(y) = -g \int d^3 \vec{y} e^{I(C)} T_{g\alpha}^\mu(y, \vec{x}) C_\mu^\alpha(y). \quad (12.19)$$

The gravitational force that acts on the mass point is

$$f_i = g \int d^3y e^{I(C)} T_{g\alpha}^\mu(y, \vec{x}) F_{i\mu}^\alpha + g \int d^3y e^{I(C)} T_{g\alpha}^\mu(y, \vec{x}) \frac{\partial}{\partial y^\mu} C_i^\alpha. \quad (12.20)$$

For quasi-static system, if we omit higher order contributions, the second term in the above relation vanish. For mass point, using the technique of Lorentz covariance analysis, we can proved that

$$P_{g\alpha} U^\mu = \gamma T_{g\alpha}^\mu, \quad (12.21)$$

where U^μ is velocity, γ is the rapidity, and $P_{g\alpha}$ is the gravitational energy-momentum. According eq.(12.18), $P_{g\alpha}$ is given by

$$P_{g\alpha} = \int d^3 \vec{y} T_{g\alpha}^0(y) = T_{g\alpha}^0. \quad (12.22)$$

Using all these relations and eq.(12.2), we get

$$\vec{f} = -ge^{I(C)} P_{g\alpha} \vec{E}^\alpha - ge^{I(C)} P_{g\alpha} \vec{v} \times \vec{B}^\alpha.$$

For quasi-static system, the dominant contribution of the above relation is

$$\vec{f} = ge^{I(C)} M \vec{E}^0 + ge^{I(C)} M \vec{v} \times \vec{B}^0, \quad (12.23)$$

where $\vec{v} = \vec{U} / \gamma$ is the velocity of the mass point. The first term of eq.(12.23) is the classical Newton's gravitational interactions. The second term of eq.(12.23) is the Lorentz force. The direction of this force is perpendicular to the direction of the motion of the mass point. When the mass point is at rest or is moving along the direction of the gravitational magnetic field, this force vanishes. Lorentz force is important for cosmology, because the rotation of galaxy will generate gravitational magnetic field and this gravitational magnetic field will affect the motion of stars and affect the large scale structure of galaxy.

4. [Origin of Terrestrial Magnetism]

It the traditional theory, it is hard to explain the origin of terrestrial magnetism. But now, using the gravitational gauge theory, it is easy to explain the origin of terrestrial magnetism. According to above discussions, we know that a rotating celestial object will give rise to gravitational magnetic field, so rotating earth will generate gravitational magnetic field around it. According to unified gravitation-electromagnetic theory[39], there is direct coupling between spin and gravitational magnetic field. In other words, electromagnetic

magnet will directly interact with gravitational magnetic field. Therefore, the origin of terrestrial magnetism is gravitational magnetic field. In other words, terrestrial magnetism and solar magnetism are not electromagnetic magnetism, but gravitational magnetism.

5. [**Negative Energy**] First, let's discuss inertial energy of pure gravitational wave. Suppose that the gravitational wave is not so strong, so the higher order contribution is very small. We only consider leading order contribution here. For pure gravitational field, we have

$$\frac{\partial \mathcal{L}_0}{\partial \partial_\mu C_\nu^\beta} = -\eta^{\mu\rho} \eta^{\nu\sigma} \eta_{2\beta\gamma} F_{\rho\sigma}^\gamma + g \eta^{\lambda\rho} \eta^{\nu\sigma} \eta_{2\beta\gamma} C_\lambda^\mu F_{\rho\sigma}^\gamma. \quad (12.24)$$

From eq.(4.32), we can get the inertial energy-momentum tensor of gravitational field in the leading order approximation, that is

$$T_{i\alpha}^\mu = e^{I(C)} [+\eta^{\mu\rho} \eta^{\nu\sigma} \eta_{2\beta\gamma} F_{\rho\sigma}^\gamma \partial_\alpha C_\nu^\beta + \delta_\alpha^\mu \mathcal{L}_0]. \quad (12.25)$$

Using eq.(12.2), Lagrangian given by eq.(4.20) can be changed into

$$\mathcal{L}_0 = \frac{1}{2} \eta_{2\alpha\beta} (\vec{E}^{\rightarrow\alpha} \cdot \vec{E}^{\rightarrow\beta} - \vec{B}^{\rightarrow\alpha} \cdot \vec{B}^{\rightarrow\beta}). \quad (12.26)$$

Space integral of time component of inertial energy-momentum tensor gives out inertial energy H_i and inertial momentum \vec{P}_i . They are

$$H_i = \int d^3 \vec{x} e^{I(C)} \left[\frac{1}{2} \eta_{2\alpha\beta} (\vec{E}^{\rightarrow\alpha} \cdot \vec{E}^{\rightarrow\beta} + \vec{B}^{\rightarrow\alpha} \cdot \vec{B}^{\rightarrow\beta}) \right], \quad (12.27)$$

$$\vec{P}_i = \int d^3 \vec{x} e^{I(C)} \eta_{2\alpha\beta} \vec{E}^{\rightarrow\alpha} \times \vec{B}^{\rightarrow\beta}. \quad (12.28)$$

In order to obtain eq.(12.27), eq.(12.3) is used. Let consider the inertial energy-momentum of gravitaional field C_μ^0 . Because,

$$\eta_{200} = -1, \quad (12.29)$$

eq.(12.27) gives out

$$H_i(C^0) = -\frac{1}{2} \int d^3 \vec{x} e^{I(C)} (\vec{E}^{\rightarrow 0} \cdot \vec{E}^{\rightarrow 0} + \vec{B}^{\rightarrow 0} \cdot \vec{B}^{\rightarrow 0}). \quad (12.30)$$

$H_i(C^0)$ is a negative quantity. It means that the inertial energy of gravitational field C_μ^0 is negative. The gravitational energy-momentum of pure gravitational gauge field is given by eq.(4.40). In leading order approximation, it is

$$\begin{aligned} T_{g\alpha}^\mu &= \eta^{\mu\rho} \eta^{\nu\sigma} \eta_{2\beta\gamma} F_{\rho\sigma}^\gamma (\partial_\alpha C_\nu^\beta) + \eta_{1\alpha}^\mu \mathcal{L}_0 \\ &\quad - \eta^{\lambda\rho} \eta^{\mu\sigma} \eta_{2\alpha\beta} \partial_\nu (C_\lambda^\nu F_{\rho\sigma}^\beta) + \eta^{\nu\rho} \eta^{\mu\sigma} \eta_{2\alpha\beta} \eta_{1\tau}^\lambda (\partial_\nu C_\lambda^\tau) F_{\rho\sigma}^\beta. \end{aligned} \quad (12.31)$$

After omitting surface terms, the gravitational energy of the system is

$$H_g = \int d^3 \vec{x} \left[\frac{1}{2} \eta_{2\alpha\beta} (\vec{E}^\alpha \cdot \vec{E}^\beta + \vec{B}^\alpha \cdot \vec{B}^\beta) - \eta^{ij} \partial_0 (C_j^0 E_i^0) \right]. \quad (12.32)$$

The gravitational energy of gravitational field C_μ^0 is,

$$H_g(C^0) = -\frac{1}{2} \int d^3 \vec{x} (\vec{E}^0 \cdot \vec{E}^0 + \vec{B}^0 \cdot \vec{B}^0 + 2\eta^{ij} \partial_0 (C_j^0 E_i^0)). \quad (12.33)$$

H_g is also negative. It means that the gravitational energy of gravitational field C_μ^0 is negative. In other words, gravitational gauge field C_μ^0 has negative gravitational energy and negative inertial energy. But, inertial mass is not equivalent to gravitational mass for pure gravitational gauge field.

6. [**Gravitational Radiation**] Because gravitational gauge field C_μ is a vector field, its dominant radiation is gravitational dipole radiation. Now, let's discuss gravitational dipole radiation. The equation of motion of gravitational gauge field is

$$\partial^\mu F_{\mu\nu}^\alpha = -g\eta_{\nu\tau} \eta_2^{\alpha\beta} T_{g\beta}^\tau. \quad (12.34)$$

For the sake of simplicity, suppose that the strength of gravitational gauge field is weak, i.e.

$$gC_\mu^\alpha \ll 1. \quad (12.35)$$

Then in leading order approximation, eq.(12.34) gives out

$$\partial^\mu (\partial_\mu C_\nu^\alpha - \partial_\nu C_\mu^\alpha) = -g\eta_{\nu\tau} \eta_2^{\alpha\beta} T_{g\beta}^\tau. \quad (12.36)$$

If we adopt Lorentz gauge

$$\partial^\mu C_\mu^\alpha = 0, \quad (12.37)$$

then eq.(12.36) is changed into

$$\partial^\mu \partial_\mu C_\nu^\alpha = -g\eta_{\nu\tau} \eta_2^{\alpha\beta} T_{g\beta}^\tau. \quad (12.38)$$

$T_{g\beta}^\tau$ is a function of space-time,

$$T_{g\beta}^\tau = T_{g\beta}^\tau(\vec{x}, t). \quad (12.39)$$

The solution to eq.(12.38) is

$$C_\nu^\alpha(\vec{x}, t) = g\eta_{\nu\tau} \eta_2^{\alpha\beta} \int \frac{T_{g\beta}^\tau(\vec{y}, t-r)}{4\pi r} d^3 \vec{y}, \quad (12.40)$$

where r is the distance between point \vec{x} and point \vec{y} ,

$$r = |\vec{x} - \vec{y}|. \quad (12.41)$$

Suppose that the object is a mass point. Then in the center of mass system of the moving particle, the solution is given by eq.(9.10), i.e.

$$C_0^0 = \frac{g}{4\pi r} P_{g0}. \quad (12.42)$$

Make a Lorentz transformation, we can get the corresponding solution in laboratory system,

$$C_\mu^\alpha = \frac{g}{4\pi M \gamma} \eta^{\alpha\beta} P_{g\mu} P_{g\beta} \frac{1}{r - \vec{v} \cdot \vec{r}}, \quad (12.43)$$

where M is the gravitational mass of the mass point, \vec{v} is the velocity of the mass point. In the above function, all variables are functions of $t' = t - r$. Suppose that the velocity of the mass point is much less than the speed of light, then we can obtain that

$$\vec{B}^\alpha = -\frac{g}{4\pi r^2} \left(\frac{d}{dt} (P_g^\alpha \vec{v}) \right) \times \vec{r}, \quad (12.44)$$

$$\vec{E}^\alpha = -\frac{g}{4\pi r^3} P_g^\alpha \vec{r} - \frac{g}{4\pi r} \dot{P}_g^\alpha \left(\frac{\vec{r}}{r} - \vec{v} \right) - \frac{g}{4\pi r^3} P_g^\alpha \vec{r} \times (\vec{r} \times \dot{\vec{v}}). \quad (12.45)$$

The first term in eq.(12.45) is the traditional Newton's gravitational field, which has no contribution on gravitational wave radiation. We will omit it in the calculation of radiation power, i.e., we use the following relation in the calculation of radiation power,

$$\vec{E}^\alpha = -\frac{g}{4\pi r} \dot{P}_g^\alpha \left(\frac{\vec{r}}{r} - \vec{v} \right) - \frac{g}{4\pi r^3} P_g^\alpha \vec{r} \times (\vec{r} \times \dot{\vec{v}}). \quad (12.46)$$

In order to calculate the radiation power, let's first determine the radiation energy flux density. Suppose that there is a space region which is denoted as Σ whose surface is denoted as Ω . The gravitational force density is denoted as $\vec{f}(x)$ and the speed of the mass point at point \vec{x} is $\vec{v}(x)$. Then in one unit time, the work that the system obtained from gravitational field is

$$\int_{\Sigma} \vec{f} \cdot \vec{v} d^3 \vec{x}. \quad (12.47)$$

Suppose that $w(x)$ is the energy density of the system. Then in one unit time, the increased energy of the system is

$$\frac{d}{dt} \int_{\Sigma} w(x) d^3 \vec{x}. \quad (12.48)$$

Supposed that the energy flux density is \vec{S} . Then in one unit time, the energy that flow into the system through the surface of the system is

$$- \oint_{\Omega} \vec{S} \cdot d\vec{\sigma} = - \int_{\Sigma} (\nabla \cdot \vec{S}) d^3 \vec{x}. \quad (12.49)$$

Energy conservation law gives out the following equation

$$- \oint_{\Omega} \vec{S} \cdot d\vec{\sigma} = \int_{\Sigma} \vec{f} \cdot \vec{v} d^3 \vec{x} + \frac{d}{dt} \int_{\Sigma} w(x) d^3 \vec{x}. \quad (12.50)$$

Because integration region is an arbitrary region, from eq.(12.50), we can obtain

$$\nabla \cdot \vec{S} + \frac{\partial w}{\partial t} = - \vec{f} \cdot \vec{v}. \quad (12.51)$$

For the sake of simplicity, suppose that the gravitational gauge field is very weak, then in leading order approximation, space-time derivative of $e^{I(C)}$ can be neglected. Using eq.(12.23), we get

$$\vec{f} \cdot \vec{v} = -g e^{I(C)} T_{g\alpha}^i E_i^\alpha. \quad (12.52)$$

Using eq.(12.9) and eq.(12.13), we get

$$\vec{f} \cdot \vec{v} = -\nabla \cdot (e^{I(C)} \vec{E}^\alpha \times \vec{B}_\alpha) - \frac{\partial}{\partial t} [\frac{1}{2} e^{I(C)} (\vec{E}^\alpha \cdot \vec{E}_\alpha + \vec{B}^\alpha \cdot \vec{B}_\alpha)]. \quad (12.53)$$

Compare eq.(12.53) with eq.(12.51), we will get

$$\vec{S} = e^{I(C)} \vec{E}^\alpha \times \vec{B}_\alpha, \quad (12.54)$$

$$w = \frac{1}{2} e^{I(C)} (\vec{E}^\alpha \cdot \vec{E}_\alpha + \vec{B}^\alpha \cdot \vec{B}_\alpha). \quad (12.55)$$

\vec{S} is the gravitational energy flux density. We use it to calculate the gravitational radiation power. Using eq.(12.44) and eq.(12.46), we can get

$$\begin{aligned} \vec{S} &= \frac{g^2}{16\pi^2 r^3} e^{I(C)} \dot{P}_g^\alpha \left[\frac{d}{dt} (P_{g\alpha} \vec{v}) (r - \vec{r} \cdot \vec{v}) \right. \\ &\quad \left. - \vec{r} \dot{P}_{g\alpha} \left(\frac{\vec{r} \cdot \vec{v}}{r} - v^2 \right) - \vec{r} P_{g\alpha} \left(\frac{\vec{r} \cdot \dot{\vec{v}}}{r} - \vec{v} \cdot \dot{\vec{v}} \right) \right] \\ &\quad + \frac{g^2}{16\pi^2 r^5} e^{I(C)} [P_{g\alpha} P_g^\alpha \vec{r} (r^2 \dot{v}^2 - (\vec{r} \cdot \dot{\vec{v}})^2) \\ &\quad + P_{g\alpha} \dot{P}_g^\alpha \vec{r} (r^2 \dot{\vec{v}} \cdot \vec{v} - (\vec{r} \cdot \dot{\vec{v}})(\vec{r} \cdot \vec{v}))]. \end{aligned} \quad (12.56)$$

The above relation can be written into another form

$$\begin{aligned}\vec{S} = & \frac{g^2}{16\pi^2 r^2} e^{I(C)} P_g^\alpha \left[\dot{\vec{v}} \cdot \frac{d}{dt} (P_{g\alpha} \vec{v}) - (\vec{n} \cdot \dot{\vec{v}}) (\vec{n} \cdot \frac{d}{dt} (P_{g\alpha} \vec{v})) \right] \vec{n} \\ & - \frac{g^2}{16\pi^2 r^2} e^{I(C)} \dot{P}_g^\alpha \left[(\vec{n} \cdot \vec{v} - 1) \frac{d}{dt} (P_{g\alpha} \vec{v}) - \vec{n} \cdot [(\vec{v} - \vec{n}) \cdot \frac{d}{dt} (P_{g\alpha} \vec{v})] \right],\end{aligned}\quad (12.57)$$

where

$$\vec{n} = \frac{\vec{r}}{r}. \quad (12.58)$$

7. **[Repulsive Force]** The classical gravitational interactions are attractive interactions. But in gravitational gauge theory, there are repulsive interactions as well as attractive interactions. The gravitational force is given by eq.(12.23). The first term corresponds to classical gravitational force. It is

$$f_i = g e^{I(C)} T_{g\alpha}^0 (\partial_i C_0^\alpha). \quad (12.59)$$

For quasi-static gravitational field, it is changed into

$$\begin{aligned}f_i &= -g e^{I(C)} P_{g\alpha} E_i^\alpha \\ &= g e^{I(C)} (M_1 E_i^0 - P_{gj} E_i^j),\end{aligned}\quad (12.60)$$

where M_1 is the gravitational mass of the mass point which is moving in gravitational field. Suppose that the gravitational field is generated by another mass point whose gravitational energy-momentum is Q_g^α and gravitational mass is M_2 . For quasi-static gravitational field, eq.(12.45) gives out

$$E_i^\alpha = -\frac{g}{4\pi r^3} Q_g^\alpha r_i \quad (12.61)$$

Substitute eq.(12.61) into eq.(12.60), we get

$$\vec{f} = e^{I(C)} \frac{g^2}{4\pi r^3} \vec{r} (-E_{1g} E_{2g} + \vec{P}_g \cdot \vec{Q}_g), \quad (12.62)$$

where E_{1g} and E_{2g} are gravitational energy of two mass point. From eq.(12.62), we can see that, if $\vec{P}_g \cdot \vec{Q}_g$ is positive, the corresponding gravitational force between two momentum is repulsive. This repulsive force is important for the stability of some celestial object. For relativistic system, all mass point moving at a high speed which is near the speed of light. Then the term $\vec{P}_g \cdot \vec{Q}_g$ has approximately the same order of magnitude as that of $E_{1g} E_{2g}$, therefore, for relativistic systems, the gravitational attractive force is not so strong as the force when all mass points are at rest.

8. **[Massive Graviton]** We have discussed pure gravitational gauge field in chapter 4. From eq.(4.20), we can see that there is no mass term of graviton in the Lagrangian. We can see that, if we introduce mass term of graviton into Lagrangian, the gravitational gauge symmetry of the action will be violated. Therefore, in that model, the graviton is massless. However, just from this Lagrangian, we can not say that there is no massive graviton in Nature. In literature [38], a new mechanism for mass generation of gauge field is proposed. The biggest advantage of this mass generation mechanism is that the mass term of gauge fields does not violate the local gauge symmetry of the Lagrangian. This mechanism is also applicable to gravitational gauge theory.

In order to introduce mass term of gravitational gauge fields, we need two sets of gravitational gauge fields simultaneously. Suppose that the first set of gauge fields is denoted as C_μ^α , and the second set of gauge fields is denoted as $C_{2\mu}^\alpha$. Under gravitational gauge transformation, they transform as

$$C_\mu(x) \rightarrow C'_\mu(x) = \hat{U}_\epsilon(x)C_\mu(x)\hat{U}_\epsilon^{-1}(x) + \frac{i}{g}\hat{U}_\epsilon(x)(\partial_\mu\hat{U}_\epsilon^{-1}(x)), \quad (12.63)$$

$$C_{2\mu}(x) \rightarrow C'_{2\mu}(x) = \hat{U}_\epsilon(x)C_{2\mu}(x)\hat{U}_\epsilon^{-1}(x) - \frac{i}{\alpha g}\hat{U}_\epsilon(x)(\partial_\mu\hat{U}_\epsilon^{-1}(x)). \quad (12.64)$$

Then, there are two gauge covariant derivatives,

$$D_\mu = \partial_\mu - igC_\mu(x), \quad (12.65)$$

$$D_{2\mu} = \partial_\mu + i\alpha gC_{2\mu}(x), \quad (12.66)$$

and two different strengths of gauge fields,

$$F_{\mu\nu} = \frac{1}{-ig}[D_\mu, D_\nu], \quad (12.67)$$

$$F_{2\mu\nu} = \frac{1}{i\alpha g}[D_{2\mu}, D_{2\nu}]. \quad (12.68)$$

The explicit forms of field strengths are

$$F_{\mu\nu} = \partial_\mu C_\nu(x) - \partial_\nu C_\mu(x) - igC_\mu(x)C_\nu(x) + igC_\nu(x)C_\mu(x), \quad (12.69)$$

$$F_{2\mu\nu} = \partial_\mu C_{2\nu}(x) - \partial_\nu C_{2\mu}(x) + i\alpha gC_{2\mu}(x)C_{2\nu}(x) - i\alpha gC_{2\nu}(x)C_{2\mu}(x). \quad (12.70)$$

The explicit forms of component strengths are

$$F_{\mu\nu}^\alpha = \partial_\mu C_\nu^\alpha - \partial_\nu C_\mu^\alpha - gC_\mu^\beta\partial_\beta C_\nu^\alpha + gC_\nu^\beta\partial_\beta C_\mu^\alpha, \quad (12.71)$$

$$F_{2\mu\nu}^\alpha = \partial_\mu C_{2\nu}^\alpha - \partial_\nu C_{2\mu}^\alpha + \alpha g C_{2\mu}^\beta \partial_\beta C_{2\nu}^\alpha - \alpha g C_{2\nu}^\beta \partial_\beta C_{2\mu}^\alpha. \quad (12.72)$$

Using eq.(12.63-64), we can obtain the following transformation properties,

$$D_\mu(x) \rightarrow D'_\mu(x) = \hat{U}_\epsilon D_\mu(x) \hat{U}_\epsilon^{-1}, \quad (12.73)$$

$$D_{2\mu}(x) \rightarrow D'_{2\mu}(x) = \hat{U}_\epsilon D_{2\mu}(x) \hat{U}_\epsilon^{-1}, \quad (12.74)$$

$$F_{\mu\nu} \rightarrow F'_{\mu\nu} = \hat{U}_\epsilon F_{\mu\nu} \hat{U}_\epsilon^{-1}, \quad (12.75)$$

$$F_{2\mu\nu} \rightarrow F'_{2\mu\nu} = \hat{U}_\epsilon F_{2\mu\nu} \hat{U}_\epsilon^{-1}, \quad (12.76)$$

$$(C_\mu + \alpha C_{2\mu}) \rightarrow (C'_\mu + \alpha C'_{2\mu}) = \hat{U}_\epsilon (C_\mu + \alpha C_{2\mu}) \hat{U}_\epsilon^{-1}. \quad (12.77)$$

The Lagrangian of the system is

$$\begin{aligned} \mathcal{L}_0 = & -\frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} \eta_{2\alpha\beta} F_{\mu\nu}^\alpha F_{\rho\sigma}^\beta - \frac{1}{4} \eta^{\mu\rho} \eta^{\nu\sigma} \eta_{2\alpha\beta} F_{2\mu\nu}^\alpha F_{2\rho\sigma}^\beta \\ & - \frac{m^2}{2(1+\alpha^2)} \eta^{\mu\nu} \eta_{2\beta\gamma} (C_\mu^\beta + \alpha C_{2\mu}^\beta) (C_\nu^\gamma + \alpha C_{2\nu}^\gamma). \end{aligned} \quad (12.78)$$

The full Lagrangian is defined by eq.(4.24) and the action is defined by eq.(4.25). It is easy to prove that the action S has local gravitational gauge symmetry. From eq.(12.78), we can see that there is mass term of gravitational gauge fields. Make a rotation,

$$C_{3\mu} = \cos\theta C_\mu + \sin\theta C_{2\mu}, \quad (12.79)$$

$$C_{4\mu} = -\sin\theta C_\mu + \cos\theta C_{2\mu}, \quad (12.80)$$

where θ is given by

$$\cos\theta = 1/\sqrt{1+\alpha^2}, \quad \sin\theta = \alpha/\sqrt{1+\alpha^2}. \quad (12.81)$$

Then, the mass term in eq.(12.78)

$$-\frac{m^2}{2} \eta^{\mu\nu} \eta_{2\beta\gamma} C_{3\mu}^\beta C_{3\nu}^\gamma. \quad (12.82)$$

So, the gravitational gauge field $C_{3\mu}$ is massive whose mass is m while gravitational gauge field $C_{4\mu}$ keeps massless.

The existence of massive graviton in Nature is very important for cosmology. Because the coupling constant for gravitational interactions and the massive graviton only take part in gravitational interactions, the massive graviton must be a relative stable particle in Nature. So, if there is massive graviton in Nature, there must be a huge amount of massive graviton in Nature and they stay at intergalactic space.

9. [**Gravitational Red Shift**] Some celestial objects, such as quasars, have great red shift. This new quantum theory of gravity will help us to understand some kinds of big red shift. Supposed that a photon which is emitted from an atom on earth has definite energy E_0 . It is known that on earth, the gravitational field is very weak, i.e.

$$g\eta_{1\alpha}^\mu C_\mu^\alpha \ll 1, \quad (12.83)$$

therefore, the factor $e^{I(C)}$ is almost 1 on earth. If the atom is not on earth, but on a celestial object which has strong gravitational field, according to eq.(5.20), eq.(6.18), and eq.(7.17), the inertial energy of the photon is

$$e^{I(C)} \cdot E_0. \quad (12.84)$$

Suppose that the gravitational mass of celestial object is M , according to eq.(9.10) and eq.(9.17), we have

$$e^{I(C)} = \exp\left(-\frac{GM}{r}\right). \quad (12.85)$$

According to eq.(12.19), at that place that the photon is emitted, the gravitational potential energy of the photon is

$$-E_0 \cdot \exp\left(-\frac{GM}{r}\right) \cdot \frac{GM}{r}. \quad (12.86)$$

When this photon arrives earth, its inertial energy is

$$E_0 \cdot \exp\left(-\frac{GM}{r}\right) \cdot \left(1 - \frac{GM}{r}\right). \quad (12.87)$$

Therefore, its gravitational red shift is

$$z = \exp\left(\frac{GM}{r}\right) \cdot \left(1 - \frac{GM}{r}\right)^{-1} - 1. \quad (12.88)$$

If the mass of the celestial object is very big and the radius of the celestial object is small, its red shift is big. As an example, suppose that $\frac{GM}{r} \sim 0.6$, its gravitational red shift is about 3.6. In other words, the gravitational red shift predicted by gravitational gauge theory is much larger than that predicted by general relativity.

10. [**Violation of Inverse Square Law And Black Hole**] It is known that classical Newton's theory of gravity predicts that gravity obeys inverse square law. According to gravitational gauge theory, the inverse square law will be

violated by intense gravitational field. According to eq.(12.65) and eq.(9.10), the gravitational force between two relative rest objects is

$$\vec{f} = -\exp\left(-\frac{GM}{r}\right) \cdot \frac{GMM_2}{r^3} \vec{r}, \quad (12.89)$$

where M and M_2 are gravitational masses of two objects. In eq.(12.89), we have supposed that M_2 is much less than M . We can clearly see that the factor $\exp\left(-\frac{GM}{r}\right)$ violates inverse square law. We can also see that, if the distance r approaches zero, the gravitational force does not approach infinity. This result is important for cosmology. If there were no the factor $\exp\left(-\frac{GM}{r}\right)$ and if the distance r approaches zero, the gravitational force would approach infinity. For black hole, there is no force that can resist gravitational force, therefore, black hole will collapse forever until it becomes a singularity in space-time. Because of the existence of the factor $\exp\left(-\frac{GM}{r}\right)$, the gravitational force does not obey inverse square law, and therefore the black hole will not collapse forever. In this point of view, black hole is not a singularity in space-time, and therefore, black hole has its own structure.

According to eq.(12.19), the gravitational potential $\phi(r)$ which is generated by an celestial object with gravitational mass M is

$$\phi(r) = -\exp\left(-\frac{GM}{r}\right) \frac{GM}{r}. \quad (12.90)$$

Suppose that the gravitational mass M is a constant. From eq.(12.90), we can see that, when distance r approaches zero or infinity, the gravitational potential approaches zero. But if distance r is a finite but non-zero, the gravitational potential is negative. There must be a definite distance R_0 where the gravitational potential reaches its minimum. The distance R_0 is given by

$$\frac{d}{dr}\phi(r)|_{r=R_0} = 0. \quad (12.91)$$

Using eq.(12.90), we can easily determine the value of R_0 ,

$$R_0 = GM. \quad (12.92)$$

From eq.(12.89), we know that, when $r = R_0$, the gravitational red shift is infinity, and therefore outside world can not see anything happens in this celestial object. In other words, if the radius of an celestial object is R_0 , it will be a black hole, and the gravitational potential reaches its minimum at the surface of the celestial object.

Suppose that there is a mass point with mass m locate at the surface of the celestial object. According to eq.(12.89), the gravitational force that acts on the mass point is

$$f(r) = \exp\left(-\frac{GM}{r}\right) \cdot \frac{GMm}{r^2}. \quad (12.93)$$

Suppose that the gravitational force $f(r)$ is strongest at distance R_1 . The distance R_1 is given by

$$\frac{d}{dr}f(r)|_{r=R_1} = 0. \quad (12.94)$$

From eq.(12.93), we get

$$R_1 = \frac{GM}{2} = \frac{R_0}{2}. \quad (12.95)$$

So, when the distance r becomes shorter than R_1 , the gravitational force will become weaker. The radius of black hole is determined by the balance where gravitational force is equivalent to pressure.

Besides, the renormalization effects will change the value of gravitational coupling constant g , and therefore affect inverse square law. This effect is a quantum effect.

11. [**Energy Generating Mechanism**] It is known that some celestial objects, such as quasar, pulsar, \dots , radiate huge amount of energy at one moment. Where is the energy comes from? We know that the gravitational field of these celestial objects are very strong, and therefore the gravitational wave radiation will also be very strong. According to gravitational gauge theory, the inertial energy of gravitational field C_μ^0 is negative. For ordinary celestial objects, their moving speed is much less than the speed of light. In this case, the dominant component of gravitational wave is C_μ^0 , therefore, gravitational wave carries negative inertial energy. It means that ordinary celestial objects obtain inertial energy through gravitational wave radiation. It is the source of part of the thermal radiation energy of these celestial objects. It is also the ultimate energy source of the whole Universe.

Gravitational wave radiation may cause disastrous consequence for black hole. Because of intense gravitational force of black hole, any positive energy can not escape from black hole, but negative energy can escape from black hole. Because gravitational gauge field C_μ^0 carries negative inertial energy and negative gravitational energy, it can escape from black hole. In other words, black hole can radiate gravitational wave, but it can not radiate electromagnetic wave. Black holes can obtain inertial energy through gravitational wave radiation,

but it can not radiate inertial energy to outside world. As a result of gravitational wave radiation, the inertial energy of black hole becomes larger and larger, its temperature becomes higher and higher, and its pressure becomes higher and higher. Then at a time, gravitational force can not resist pressure from inside of the block hole. It will burst and release huge amount of inertial energy at a relative short time.

12. [**Dark Matter**] Dark matter is an important problem in cosmology. In gravitational gauge field theory, the following effects are helpful to solve this problem: 1) The existence of massive graviton will contribute some to dark matter. 2) If the gravitational magnetic field is strong inside a celestial system, the Lorentz force will provide additional centripetal force for circular motion of a celestial object. 3) The existence of the factor $e^{I(C)}$ violate inverse square law of classical gravity. Besides, there are a lot of other possibilities which is widely discussed in literature. I will not list them here, for they have nothing to do with the gravitational gauge theory.

13. [**Particle Accelerating Mechanism**] It is known that there are some cosmic rays which have extremely high energy. It is hard to understand why some cosmic particles can have energy as high as 10^{21} eV. Gravitational gauge theory gives a possible explanation for this phenomenon. According to gravitational gauge theory, there are strong gravitational magnetic field inside a galaxy. A cosmic particle will make circular motion around the center of galaxy under Lorentz force which is provided by gravitational magnetic field and classical Newton's gravitational force. These cosmic particles will radiate gravitational wave when they moving in gravitational field. Because gravitational wave carries negative inertial energy, cosmic particles will be accelerated when they radiate gravitational wave. If this explanation is correct, most cosmic particles which have extremely high energy will come from the center of a galaxy.

14. [**Equivalence Principle**] Equivalence principle is one of the most important foundations of general relativity, but it is not a logic starting point of gravitational gauge field theory. The logic starting point of gravitational gauge field theory is gauge principle. However, one important inevitable result of gauge principle is that gravitational mass is not equivalent to inertial mass. The origin of violation of equivalence principle is gravitational field. If there were no gravitational field, equivalence principle would strictly hold. But if gravitational field is strong, equivalence principle will be strongly violated. For some celestial objects which have strong gravitational field, such as quasar

and black hole, their gravitational mass will be higher than their inertial mass. But on earth, the gravitational field is very weak, so the equivalence principle almost exactly holds. We need to test the validity of equivalence principle in astrophysics experiments.

15. [**Violation of the Second Law of Thermodynamics**]If we treat the whole universe as an isolated system, according to the second law of thermodynamics, our Universe will finally go to the completely statistical equilibrium, which is main point of the theory of heat death. If we consider the influence of gravitational interactions, second law of thermodynamics no longer holds, for an object can obtain energy through radiating gravitational wave. For example, black hole can obtain energy from outside world through gravitational wave radiation, though the temperature of black hole is much higher than outside world. In this meaning, black hole is a perpetual motion machine of the Universe, and because of the existence of this perpetual motion machine, our Universe will not go to the state of heat death.

16. [**New Energy Source**]It is known that one of the biggest problem for the development of civilization of human kind in future is the energy crisis. However, gravitational gauge field theory provides an everlasting energy source for both human kind and the whole universe. Two hundred years ago, human kind does not know how to utilize electric energy, but now most energy comes from electric energy. It is believed that, human kind will eventually know how to utilize gravitational energy in future. If so, most energy will come from perpetual motion machine in future.

13 Summary

In this paper, we have discussed a completely new quantum gauge theory of gravity. Finally, we give a simple summary to the whole theory.

1. In leading order approximation, the gravitational gauge field theory gives out classical Newton's theory of gravity.
2. In first order approximation and for vacuum, the gravitational gauge field theory gives out Einstein's general theory of relativity.
3. Gravitational gauge field theory is a renormalizable quantum theory.

References

- [1] Isaac Newton, *Mathematical Principles of Natural Philosophy*, (Cambridge University Press, 1934) .
- [2] Albert Einstein, *Annalen der Phys.*, **49** (1916) 769 .
- [3] Albert Einstein, *Zeits. Math. und Phys.* **62** (1913) 225.
- [4] H.Weyl, in *Raum. Zeit. Materia*, 3rd ed. (Springer-Verlag, Berlin, 1920)
- [5] H.Weyl, *Gravitation and Elektrizitat*. Sitzungsber. Akademie der Wissenschaften Berlin, 465-480(1918). Siehe auch die *Gesammelten Abhandlungen*. 6 Vols. Ed. K.Chadrsekharan, Springer-Verlag.
- [6] V.Fock, *Zeit. f. Physik*, **39** (1927) 226.
- [7] H.Weyl, *Zeit. f. Physik*, **56** (1929) 330.
- [8] W.Pauli, *Handbuch der physik, Geiger and scheel, 2, Aufl.*, Vol. 24, Teil **1** pp. 83-272.
- [9] C.N.Yang, R.L.Mills, *Phys Rev* **96** (1954) 191 .
- [10] S.Glashow, *Nucl Phys* **22**(1961) 579 .
- [11] S.Weinberg, *Phys Rev Lett* **19** (1967) 1264 .
- [12] A.Salam, in *Elementary Particle Theory*, eds.N.Svartho . Forlag, Stockholm,1968).
- [13] Albert Einstein, "Naeherungsweise Integration der Feldgleichungen der Gracitation", *Preussische Akademie der Wissenschaften (Berlin) Sitzungsberichte*, pg 688 (1916) .
- [14] G.Amelino-Camelia, *Nature*, **408** (2000) 661-664
- [15] M.Tegmark and J.A.Wheeler, *Sci.Am.* **284** (2001) 68-75
- [16] R.Utiyama, *Phys.Rev.***101** (1956) 1597.
- [17] A.Brodsky, D.Ivanenko and G. Sokolik, *JETPH* 41 (1961)1307; *Acta Phys.Hung.* **14** (1962) 21.
- [18] T.W.Kibble, *J.Math.Phys.* **2** (1961) 212.
- [19] D.Ivanenko and G.Sardanashvily, *Phys.Rep.* **94** (1983) 1.

- [20] F.W.Hehl, J.D.McCrea, E.W.Mielke and Y.Ne'eman Phys.Rep. **258** (1995) 1-171
- [21] F.W.Hehl, P. Von Der Heyde, G.D.Kerlick, J.M.Nester Rev.Mod.Phys. **48** (1976) 393-416
- [22] F.Gronwald and F.W.Hehl, *On the Gauge Aspects of Gravity*, gr-qc/9602013
- [23] F.Brandt, Phys.Rev. D64 (2001) 065025
- [24] Rolando Gaitan D., *A possible gauge formulation for gravity?*, gr-qc/0109079
- [25] M.Botta Cantcheef, *General Relativity as a (constrained) Yang-Mills's theory*, gr-qc/0010080
- [26] Ning Wu, Zhan Xu and Dahua Zhang, "Differential Geometrical Formulation of Gravitational Gauge Theory of Gravity" (in preparation)
- [27] Ning Wu, (in preparation).
- [28] L.Fadeev and V.N.Popov, Phys. Letters, **25B** (1967),29
- [29] G. 't-Hooft, Nucl. Phys. **B33** (1971) 173.
- [30] G. 't-Hooft and M. Veltman, Nucl. Phys. **B50** (1972) 318.
- [31] B.W.Lee and J. Zinn-Justin, Phys. Rev. **D5** (1972) 3121, 3137.
- [32] B.W.Lee and J. Zinn-Justin, Phys. Rev. **D7** (1973) 1049.
- [33] B.W.Lee and J. Zinn-Justin, Phys. Rev. **D9** (1974) 933.
- [34] J. Zinn-Justin, *Renormalization of Gauge Theories, Lectures at the 1974 Bonn int. Summer Inst. for Theoretical physics.*
- [35] B.W.Lee *in Methods in Field Theory* (North-Holland, 1976)
- [36] G. 't-Hooft and M. Veltman, Nucl. Phys. **B44** (1972) 189.
- [37] G. 't-Hooft and M. Veltman, *Diagram* (CERN Theoretical Studies Division 1973).
- [38] Ning Wu, Commun.Theor.Phys., **36**(2001) 169.
- [39] Ning Wu, "Unified Theory of Fundamental Interactions" (In preparation)